

Solution of Different Types of Differential Equations

4 Marks Questions

1. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$. All India 2014

Given differential equation is

$$\begin{aligned}\frac{dy}{dx} &= 1 + x + y + xy \\ \Rightarrow \frac{dy}{dx} &= 1(1+x) + y(1+x) \\ \Rightarrow \frac{dy}{dx} &= (1+x)(1+y) \quad \dots(i) \quad (1)\end{aligned}$$

On separating variables, we get

$$\frac{1}{(1+y)} dy = (1+x) dx \quad \dots(ii)$$

On integrating both sides of Eq. (ii), we get

$$\int \frac{1}{1+y} dy = \int (1+x) dx$$

$$\Rightarrow \log|1+y| = x + \frac{x^2}{2} + C \quad \dots(\text{iii}) \quad (1)$$

Also, given that $y = 0$, when $x = 1$.

On substituting $x = 1, y = 0$ in Eq. (iii), we get

$$\log|1+0| = 1 + \frac{1}{2} + C \Rightarrow C = -\frac{3}{2} \quad [\because \log 1 = 0] \quad (1)$$

Now, on substituting the value of C in Eq. (iii), we get

$$\log|1+y| = x + \frac{x^2}{2} - \frac{3}{2}$$

which is the required particular solution of given differential equation. (1)

2. Find the particular solution of the differential

equation $x \frac{dy}{dx} - y + x \operatorname{cosec} \left(\frac{y}{x} \right) = 0$ or

$$\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec} \left(\frac{y}{x} \right) = 0, \text{ given that } y = 0, \text{ when}$$

$$x = 1. \quad \text{All India 2014C, 2011; Delhi 2009}$$

Given differential equation is

$$x \frac{dy}{dx} - y + x \operatorname{cosec} \left(\frac{y}{x} \right) = 0$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec} \left(\frac{y}{x} \right) = 0$$

Above equation can be written as

$$\frac{dy}{dx} = \frac{y}{x} - \operatorname{cosec} \left(\frac{y}{x} \right) \quad \dots(i)$$

which is a homogeneous differential equation.

On putting $y = vx$,

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \text{ in Eq. (i), we get}$$

$$v + x \frac{dv}{dx} = \frac{vx}{x} - \operatorname{cosec} \left(\frac{vx}{x} \right)$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \operatorname{cosec} v$$

$$\Rightarrow x \frac{dv}{dx} = -\operatorname{cosec} v \Rightarrow \frac{dv}{\operatorname{cosec} v} = \frac{-dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{dv}{\operatorname{cosec} v} = \int -\frac{dx}{x}$$

$$\Rightarrow \int \sin v \, dv = \int -\frac{dx}{x} \quad \left[\because \frac{1}{\operatorname{cosec} v} = \sin v \right]$$

$$\Rightarrow -\cos v = -\log |x| + C$$

$$\left[\because \int \sin x \, dx = -\cos x + C \right]$$

$$\text{and } \int \frac{1}{x} \, dx = \log |x| + C$$

On putting $v = \frac{y}{x}$, we get

$$-\cos \frac{y}{x} = -\log|x| + C$$

$$\Rightarrow \cos \frac{y}{x} = +(\log|x| - C)$$

$$\Rightarrow \frac{y}{x} = \cos^{-1}(\log|x| - C)$$

$$\Rightarrow y = x \cos^{-1}(\log|x| - C) \quad \dots(ii) \quad (1\frac{1}{2})$$

Also, given that $x = 1$ and $y = 0$.

On putting above values in Eq. (ii), we get

$$0 = 1 \cos^{-1}(\log|1| - C)$$

$$\Rightarrow \cos 0^\circ = 0 - C$$

$$\Rightarrow 1 = 0 - C$$

$$\Rightarrow C = -1$$

$$\therefore y = x \cos^{-1}(\log|x| + 1) \quad (1\frac{1}{2})$$

which is required solution.

3. Solve the differential equation

$$x \log x \frac{dy}{dx} + y = \frac{2}{x} \log x. \text{ Foreign 2014; Delhi 2009}$$

Given differential equation is

$$(x \log x) \cdot \frac{dy}{dx} + y = \frac{2}{x} \log x$$

On dividing both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2 \log x}{x^2 \log x} = \frac{2}{x^2} \quad \dots(i)$$

which is a linear differential equation of first order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x \log x} \text{ and } Q = \frac{2}{x^2} \quad (1)$$

$$\therefore \text{IF} = e^{\int \frac{1}{x \log x} dx} = e^{\log \log x}$$

$$\left[\begin{array}{l} \text{for } \int \frac{1}{x \log x} dx \Rightarrow \text{put } \log x = t \Rightarrow \frac{1}{x} dx = dt \\ \therefore \int \frac{1}{t} dt = \log |t| = \log |\log x| \end{array} \right]$$
$$\Rightarrow \text{IF} = \log x \quad [\because e^{\log x} = x] \quad (1)$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C \quad \dots(iii)$$

On putting IF = $\log x$ and $Q = \frac{2}{x^2}$ in Eq. (iii),

we get

$$y \log x = \int \frac{2}{x^2} \log x \, dx$$

$$\Rightarrow y \log x = \log x \int \frac{2}{x^2} \, dx - \int \left(\frac{d}{dx} (\log x) \cdot \int \frac{2}{x^2} \, dx \right) dx$$

[using integration by parts]

$$\Rightarrow y \log x = \log x \cdot 2 \left(-\frac{1}{x} \right) - \int \frac{1}{x} \cdot 2 \left(-\frac{1}{x} \right) dx \quad (1)$$

$$\Rightarrow y \log x = -\frac{2}{x} \log x - \int \frac{2}{x} \left(-\frac{1}{x} \right) dx$$

$$\Rightarrow y \log x = -\frac{2}{x} \log x + \int \frac{2}{x^2} \, dx$$

$$\therefore y \log x = -\frac{2}{x} \log x - \frac{2}{x} + C \quad (1)$$

which is the required solution.

4. Find the general solution of the differential equation $(x - y) \frac{dy}{dx} = x + 2y$.

Delhi 2014C; All India 2010

Given differential equation is

$$(x - y) \frac{dy}{dx} = x + 2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{x + 2y}{x - y} \quad \dots(i) \quad (1)$$

which is a homogeneous equation.

On putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad \dots(ii)$$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{x + 2vx}{x - vx} = \frac{1 + 2v}{1 - v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + 2v}{1 - v} - v = \frac{1 + 2v - v + v^2}{1 - v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1 + v + v^2}{1 - v}$$

$$\Rightarrow \frac{1 - v}{v^2 + v + 1} dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{1 - v}{v^2 + v + 1} dv = \int \frac{dx}{x} \quad (1)$$

$$\Rightarrow I = \log|x| + C \quad \dots(iii)$$

where, $I = \int \frac{1 - v}{v^2 + v + 1} dv$

Let $1 - v = A \cdot \frac{d}{dv}(v^2 + v + 1) + B$

$$\Rightarrow 1 - v = A(2v + 1) + B$$

On comparing coefficients of v and constant term from both sides, we get

$$2A = -1 \Rightarrow A = -\frac{1}{2} \quad \text{and} \quad A + B = 1$$

$$\Rightarrow -\frac{1}{2} + B = 1 \Rightarrow B = 1 + \frac{1}{2} \Rightarrow B = \frac{3}{2}$$

So, we write $1 - v = -\frac{1}{2}(2v + 1) + \frac{3}{2}$

$$\text{Then, } I = \int \frac{-\frac{1}{2}(2v+1) + \frac{3}{2}}{v^2 + v + 1} dv$$

$$\Rightarrow I = -\frac{1}{2} \int \frac{2v+1}{v^2 + v + 1} dv + \frac{3}{2} \int \frac{dv}{v^2 + v + 1}$$

$$\Rightarrow I = -\frac{1}{2} \log|v^2 + v + 1| + \frac{3}{2} \int \frac{dv}{v^2 + v + 1 + \frac{1}{4} - \frac{1}{4}}$$

$$\left[\begin{array}{l} \because \int \frac{2v+1}{v^2 + v + 1} dv \Rightarrow \text{put } v^2 + v + 1 = t \\ \qquad \qquad \qquad (2v+1) dv = dt \\ \therefore \int \frac{dt}{t} = \log|t| + c = \log|v^2 + v + 1| + c \end{array} \right]$$

$$\Rightarrow I = -\frac{1}{2} \log|v^2 + v + 1| + \frac{3}{2} \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 + \frac{3}{4}} \quad (1)$$

$$\Rightarrow I = -\frac{1}{2} \log|v^2 + v + 1| + \frac{3}{2} \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow I = -\frac{1}{2} \log|v^2 + v + 1|$$

$$+ \frac{3}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{v + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C$$

$$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

$$\Rightarrow I = -\frac{1}{2} \log|v^2 + v + 1| + \frac{3}{\sqrt{3}} \tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) + C$$

On putting $v = \frac{y}{x}$, we get

$$I = -\frac{1}{2} \log\left|\frac{y^2}{x^2} + \frac{y}{x} + 1\right| + \sqrt{3} \tan^{-1}\left(\frac{\frac{2y}{x} + 1}{\sqrt{3}}\right) + C$$

$$\left[\because y = vx \therefore v = \frac{y}{x} \right]$$

$$\Rightarrow I = -\frac{1}{2} \log\left|\frac{y^2 + xy + x^2}{x^2}\right| + \sqrt{3} \tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) + C$$

On putting the value of I in Eq. (iii), we get

$$-\frac{1}{2} \log\left|\frac{y^2 + xy + x^2}{x^2}\right| + \sqrt{3} \tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) = \log|x| + C$$

which is the required solution. (1)

5. Find the particular solution of the differential equation $\left\{x \sin^2\left(\frac{y}{x}\right) - y\right\} dx + x dy = 0$, given

that $y = \frac{\pi}{4}$, when $x = 1$. All India 2014C

Given differential equation is

$$\left[x \sin^2\left(\frac{y}{x}\right) - y \right] dx + x dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \quad \dots(i)$$

which is a homogeneous differential equation.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{x dv}{dx}$ in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{vx - x \sin^2\left(\frac{vx}{x}\right)}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = v - \sin^2 v \Rightarrow x \frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \operatorname{cosec}^2 v \, dv = -\frac{dx}{x} \quad \mathbf{(1)}$$

On integrating both sides, we get

$$\int \operatorname{cosec}^2 v \, dv + \int \frac{dx}{x} = 0$$

$$\Rightarrow -\cot v + \log|x| = C$$

$$\Rightarrow -\cot\left(\frac{y}{x}\right) + \log|x| = C \quad \left[\because v = \frac{y}{x} \right] \quad \dots(ii)$$

(1)

Also, given that $y = \frac{\pi}{4}$, when $x = 1$.

On putting $x = 1$ and $y = \frac{\pi}{4}$ in Eq. (ii), we get

$$-\cot\left(\frac{\pi}{4}\right) + \log 1 = C$$

$$\Rightarrow C = -1 \quad \left[\because \cot \frac{\pi}{4} = 1 \right] \text{ (1)}$$

On putting this value of C in Eq. (ii), we get

$$-\cot\left(\frac{y}{x}\right) + \log|x| = 1$$

$$\Rightarrow 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

which is the required particular solution of given differential equation. **(1)**

6. Find the particular solution of the differential equation

$$\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y \cos y}, \text{ given that } y = \frac{\pi}{2}, \text{ when}$$

$$x = 1.$$

Delhi 2014

Given differential equation is

$$\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos y}$$

On separating the variables, we get

$$\begin{aligned} (\sin y + y \cos y) dy &= x(2 \log x + 1) dx \\ \Rightarrow \sin y dy + y \cos y dy &= 2x \log x dx + x dx \quad \text{(1)} \end{aligned}$$

On integrating both sides, we get

$$\begin{aligned} \int \sin y dy + \int y \cos y dy &= 2 \int x \log x dx + \int x dx \\ &= 2 \int x \log x dx + \int x dx \\ \Rightarrow -\cos y + \left[y \int \cos y dy \right. \\ &\quad \left. - \int \left\{ \frac{d}{dy} (y) \int \cos y dy \right\} dy \right] \\ &= 2 \left[\log x \int x dx - \int \left\{ \frac{d}{dx} (\log x) \int x dx \right\} dx \right] + \frac{x^2}{2} \end{aligned} \quad \text{(1)}$$

$$\begin{aligned} \Rightarrow -\cos y + y \sin y - \int \sin y dy \\ &= 2 \left[\frac{x^2}{2} \log x - \int \left\{ \frac{1}{x} \frac{x^2}{2} \right\} dx \right] + \frac{x^2}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow -\cos y + y \sin y + \cos y \\ &= x^2 \log x - \int x dx + \frac{x^2}{2} \\ \Rightarrow y \sin y &= x^2 \log x - \frac{x^2}{2} + \frac{x^2}{2} + C \end{aligned}$$

$$\Rightarrow y \sin y = x^2 \log x + C \quad \dots(i) \quad (1)$$

Also, given that $y = \frac{\pi}{2}$, when $x = 1$.

On putting $y = \frac{\pi}{2}$ and $x = 1$ in Eq. (i), we get

$$\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) = (1)^2 \log(1) + C$$

$$\Rightarrow C = \frac{\pi}{2} \quad \left[\because \sin \frac{\pi}{2} = 1, \log 1 = 0 \right]$$

On substituting the value of C in Eq. (i), we get

$$y \sin y = x^2 \log x + \frac{\pi}{2}$$

which is the required particular solution. **(1)**

7. Solve the following differential equation

$$(x^2 - 1) \frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}.$$

Delhi 2014; All India 2014C

💡 Firstly, divide the given differential equation by $(x^2 - 1)$ to convert it into the form of linear differential equation and then solve it.

Given differential equation is

$$(x^2 - 1) \frac{dy}{dx} + 2xy = \frac{2}{x^2 - 1}$$

On dividing both sides by $(x^2 - 1)$, we get

$$\frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{2}{(x^2 - 1)^2}$$

which is a linear differential equation. (1)

On comparing with the form $\frac{dy}{dx} + Py = Q$, we

get
$$P = \frac{2x}{x^2 - 1}, Q = \frac{2}{(x^2 - 1)^2}$$

\therefore
$$\text{IF} = e^{\int \frac{2x}{x^2 - 1} dx} \quad (1)$$

$$= e^{\log|x^2 - 1|} = x^2 - 1$$

$$\left[\begin{array}{l} \text{put } x^2 - 1 = t \Rightarrow 2x dx = dt \text{ in } \int \frac{2x}{x^2 - 1} dx, \text{ then} \\ \int \frac{2x}{x^2 - 1} dx = \int \frac{1}{t} dt = \log t = \log(x^2 - 1) \end{array} \right]$$

Hence, the required general solution is

$$y \cdot \text{IF} = \int Q \times \text{IF} dx + C$$

$$\Rightarrow y(x^2 - 1) = \int \frac{2}{(x^2 - 1)^2} \times (x^2 - 1) dx + C \quad (1)$$

$$\Rightarrow y(x^2 - 1) = \int \frac{2}{x^2 - 1} dx + C$$

$$\Rightarrow y(x^2 - 1) = \log \left| \frac{x-1}{x+1} \right| + C$$

$$\left[\because \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$

which is the required differential equation. **(1)**

- 8.** Find the particular solution of the differential equation $e^x \sqrt{1-y^2} dx + \frac{y}{x} dy = 0$, given that

$$y = 1, \text{ when } x = 0.$$

Delhi 2014

Given differential equation is

$$e^x \sqrt{1-y^2} dx + \frac{y}{x} dy = 0$$

$$\Rightarrow e^x \sqrt{1-y^2} dx = \frac{-y}{x} dy$$

On separating the variables, we get

$$\frac{-y}{\sqrt{1-y^2}} dy = x e^x dx \quad (1)$$

On integrating both sides, we get

$$\int \frac{-y}{\sqrt{1-y^2}} dy = \int x e^x dx$$

On putting $1-y^2 = t \Rightarrow -y dy = \frac{dt}{2}$ in LHS, we get

$$\int \frac{1}{2\sqrt{t}} dt = \int x e^x dx$$

$$\Rightarrow \frac{1}{2} [2\sqrt{t}] = x \int e^x dx - \int \left[\frac{d}{dx}(x) \int e^x dx \right] dx$$

$$\Rightarrow \sqrt{1-y^2} = x e^x - \int e^x dx \quad [:\because t = 1-y^2] \quad (1)$$

$$\Rightarrow \sqrt{1-y^2} = x e^x - e^x + C \quad \dots(i)$$

Also, given that $y = 1$, when $x = 0$

On putting $y = 1$ and $x = 0$ in Eq. (i), we get

$$\sqrt{1-1} = 0 - e^0 + C$$

$$\Rightarrow C = 1 \quad [:\because e^0 = 1] \quad (1)$$

On substituting the value of C in Eq. (i), we get

$$\sqrt{1-y^2} = x e^x - e^x + 1$$

which is the required particular solution of given differential equation. (1)

9. Solve the following differential equation

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2 y^2 = 0. \quad \text{Delhi 2014}$$



Firstly, separate the variables, then integrate by using integration by parts.

Given differential equation is

$$\operatorname{cosec} x \log y \frac{dy}{dx} + x^2 y^2 = 0 \quad \dots(i)$$

It can be rewritten as

$$\operatorname{cosec} x \log y \frac{dy}{dx} = -x^2 y^2$$

On separating the variables, we get

$$\frac{\log y}{y^2} dy = \frac{-x^2}{\operatorname{cosec} x} dx$$

On integrating both sides, we get

$$\int \frac{\log y}{y^2} dy = - \int \frac{x^2}{\operatorname{cosec} x} dx \Rightarrow I_1 = I_2 \dots(ii)$$

(1)

$$\text{where, } I_1 = \int \frac{\log y}{y^2} dy$$

$$\text{Put } \log y = t \Rightarrow y = e^t, \text{ then } \frac{dy}{y} = dt$$

$$\therefore I_1 = \int t e^{-t} dt$$

$$= t \int e^{-t} dt - \int \left[\frac{d}{dt}(t) \int e^{-t} dt \right] dt$$

$$= -t e^{-t} - \int (-e^{-t}) dt$$

$$= -t e^{-t} + \int e^{-t} dt = -t e^{-t} - e^{-t} + C_1$$

$$= -\frac{\log y}{y} - \frac{1}{y} + C_1 \quad \dots(iii) \quad (1)$$

$$\left[\because t = \log y \text{ and } e^{-t} = \frac{1}{y} \right]$$

$$\begin{aligned}
 \text{and } I_2 &= -\int \frac{x^2}{\operatorname{cosec} x} dx \\
 &= -\int x^2 \sin x dx \\
 &= -x^2 \int \sin x dx - \int \left[\frac{d}{dx}(x^2) \int \sin x dx \right] dx \\
 &= -x^2 (-\cos x) - \int [2x(-\cos x)] dx \\
 &= x^2 \cos x + 2 \int x \cos x dx \\
 &= x^2 \cos x + 2 \left[x \int \cos x dx \right. \\
 &\quad \left. - \int \left\{ \frac{d}{dx}(x) \int \cos x dx \right\} dx \right] \\
 &= x^2 \cos x + 2 [x \sin x - \int \sin x dx] \\
 &= x^2 \cos x + 2x \sin x + 2 \cos x + C_2 \quad \dots \text{(iv)}
 \end{aligned}$$

(1)

On putting the values of I_1 and I_2 from Eqs.(iii) and (iv) in Eq. (ii), we get

$$\begin{aligned}
 -\frac{\log y}{y} - \frac{1}{y} + C_1 &= x^2 \cos x + 2x \sin x \\
 &\quad + 2 \cos x + C_2 \\
 \Rightarrow -\frac{(1 + \log y)}{y} &= x^2 \cos x + 2x \sin x \\
 &\quad + 2 \cos x + C_2 - C_1 \\
 \Rightarrow -\frac{(1 + \log y)}{y} &= x^2 \cos x + 2x \sin x \\
 &\quad + 2 \cos x + C
 \end{aligned}$$

where, $C = C_2 - C_1$

which is the required solution of given differential equation. **(1)**

- 10.** Find the particular solution of the differential equation $x(1 + y^2) dx - y(1 + x^2) dy = 0$, given that $y = 1$, when $x = 0$. All India 2014

Given differential equation is

$$\begin{aligned}x(1 + y^2) dx - y(1 + x^2) dy &= 0 \\ \Rightarrow x(1 + y^2) dx &= y(1 + x^2) dy\end{aligned}$$

On separating the variables, we get

$$\frac{y}{(1 + y^2)} dy = \frac{x}{(1 + x^2)} dx \quad (1)$$

On integrating both sides, we get

$$\begin{aligned}\int \frac{y}{1 + y^2} dy &= \int \frac{x}{(1 + x^2)} dx \\ \Rightarrow \frac{1}{2} \log |1 + y^2| &= \frac{1}{2} \log |1 + x^2| + C \quad \dots(i)\end{aligned}$$

$$\left[\begin{array}{l} \text{let } 1 + y^2 = u \Rightarrow 2y dy = du, \\ \text{then } \int \frac{y}{1 + y^2} dy = \int \frac{1}{2u} du = \frac{1}{2} \log |u| \\ \text{and let } 1 + x^2 = v \Rightarrow 2x dx = dv, \\ \text{then } \int \frac{x}{1 + x^2} dx = \frac{1}{2} \int \frac{1}{v} dv = \frac{1}{2} \log |v| \end{array} \right]$$

Also, given that $y = 1$, when $x = 0$. (1)

On substituting the values of x and y in Eq. (i), we get

$$\begin{aligned}\frac{1}{2} \log |1 + (1)^2| &= \frac{1}{2} \log |1 + (0)^2| + C \\ \Rightarrow \frac{1}{2} \log 2 &= C \quad [\because \log 1 = 0]\end{aligned}$$

On putting $C = \frac{1}{2} \log 2$ in Eq. (i), we get

$$\frac{1}{2} \log |1 + y^2| = \frac{1}{2} \log |1 + x^2| + \frac{1}{2} \log 2$$

$$\Rightarrow \log |1+y^2| = \log |1+x^2| + \log 2 \quad (1)$$

$$\Rightarrow \log |1+y^2| - \log |1+x^2| = \log 2$$

$$\Rightarrow \log \left| \frac{1+y^2}{1+x^2} \right| = \log 2 \left[\because \log m - \log n = \log \frac{m}{n} \right]$$

$$\Rightarrow \frac{1+y^2}{1+x^2} = 2$$

$$\Rightarrow 1+y^2 = 2+2x^2 \Rightarrow y^2 - 2x^2 - 1 = 0$$

which is the required particular solution of given differential equation. (1)

11. Find the particular solution of the differential equation $\log \left(\frac{dy}{dx} \right) = 3x + 4y$ equation, given

that $y = 0$, when $x = 0$.

All India 2014

Given differential equation is

$$\log \left(\frac{dy}{dx} \right) = 3x + 4y$$

$$\Rightarrow \frac{dy}{dx} = e^{3x+4y}$$

[$\because \log m = n \Rightarrow e^n = m$]

$$\Rightarrow \frac{dy}{dx} = e^{3x} \cdot e^{4y} \quad (1)$$

On separating the variables, we get

$$\frac{1}{e^{4y}} dy = e^{3x} dx$$

On integrating both sides, we get

$$\int e^{-4y} dy = \int e^{3x} dx$$

$$\Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C \quad \dots(i) (1)$$

Also, given that $y = 0$, when $x = 0$.

On putting $y = 0$ and $x = 0$ in Eq. (i), we get

$$\frac{e^{-4(0)}}{-4} = \frac{e^{3(0)}}{3} + C$$

$$\Rightarrow -\frac{1}{4} = \frac{1}{3} + C \quad [\because e^{-0} = e^0 = 1]$$

$$\Rightarrow C = -\frac{1}{4} - \frac{1}{3}$$

$$\therefore C = \frac{-7}{12} \quad (1)$$

On substituting the value of C in Eq. (i), we get

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12}$$

which is the required particular solution of given differential equation. (1)

12. Solve the differential equation

$$(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}. \quad \text{All India 2014}$$

Given differential equation is

$$(1+x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$$

On dividing both sides by $(1+x^2)$, we get

$$\frac{dy}{dx} + \frac{1}{(1+x^2)} y = \frac{e^{\tan^{-1} x}}{1+x^2}$$

It is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

On comparing, we get

$$P = \frac{1}{1+x^2} \text{ and } Q = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$$\begin{aligned} \therefore \text{IF} &= e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x} \\ &\left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x \right] \quad (1) \end{aligned}$$

Then, required solution is

$$(y \cdot \text{IF}) = \int (Q \cdot \text{IF}) dx + C$$

$$\therefore y e^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x} \cdot e^{\tan^{-1} x}}{1+x^2} dx + C$$

$$\Rightarrow y e^{\tan^{-1} x} = \int \frac{e^{2 \tan^{-1} x}}{1+x^2} dx + C$$

$$\Rightarrow y e^{\tan^{-1} x} = I + C \quad \dots(i) \quad (1)$$

$$\text{where, } I = \int \frac{e^{2 \tan^{-1} x}}{1+x^2} dx$$

$$\text{Put } \tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$$

$$\therefore I = \int e^{2t} dt$$

$-2t \qquad -2 \tan^{-1} x$

$$\Rightarrow I = \frac{e^{-x}}{2} \Rightarrow I = \frac{e^{-x}}{2} \quad (1)$$

On putting the value of I in Eq. (i), we get

$$y e^{\tan^{-1} x} = \frac{e^{2 \tan^{-1} x}}{2} + C$$

which is the required general solution of given differential equation. (1)

13. Find a particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given that

$$y = 0, \text{ when } x = \frac{\pi}{3}.$$

Foreign 2014

Given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

which is a linear differential equation of the

form $\frac{dy}{dx} + Py = Q$.

On comparing, we get

$$P = 2 \tan x \text{ and } Q = \sin x$$

$$\therefore \text{IF} = e^{2 \int \tan x \, dx} = e^{2 \log |\sec x|} \quad (1)$$

$$= e^{\log \sec^2 x} \quad [\because m \log n = \log n^m]$$

$$= \sec^2 x \quad [\because e^{\log x} = x]$$

The general solution is given by

$$Y \cdot \text{IF} = \int Q \times \text{IF} \, dx + C \quad \dots(i) \quad (1)$$

$$\Rightarrow y \sec^2 x = \int (\sin x \cdot \sec^2 x) \, dx + C$$

$$\Rightarrow y \sec^2 x = \int \sin x \cdot \frac{1}{\cos^2 x} \, dx + C$$

$$\Rightarrow y \sec^2 x = \int \tan x \sec x \, dx + C$$

$$\Rightarrow y \sec^2 x = \sec x + C \quad \dots(ii)$$

Also, given that $y = 0$, when $x = \frac{\pi}{3}$. On putting

$y = 0$ and $x = \frac{\pi}{3}$ in Eq. (ii), we get

$$0 \times \sec^2 \frac{\pi}{3} = \sec \frac{\pi}{3} + C$$

$$\Rightarrow 0 = 2 + C \Rightarrow C = -2 \quad (1)$$

On putting the value of C in Eq. (ii), we get

$$y \sec^2 x = \sec x - 2$$

$$\Rightarrow y = \cos x - 2 \cos^2 x$$

which is the required solution of the given differential equation. (1)

14. Solve the following differential equation

$$x \cos \left(\frac{y}{x} \right) \frac{dy}{dx} = y \cos \left(\frac{y}{x} \right) + x; \quad x \neq 0.$$

All India 2014C

Given differential equation is

$$x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x \quad \dots(i)$$

which is a homogeneous differential equation.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$ in

Eq. (i), we get

$$x \cos v \left[v + x \frac{dv}{dx} \right] = vx \cos v + x$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{x(v \cos v + 1)}{x \cos v} \quad (1)$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + 1}{\cos v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + 1 - v \cos v}{\cos v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1}{\cos v} \Rightarrow \cos v \, dv = \frac{dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \cos v \, dv = \int \frac{dx}{x}$$

$$\Rightarrow \sin v = \log x + C \quad (1)$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \log x + C \left[\because y = vx \Rightarrow v = \frac{y}{x} \right]$$

which is the required solution of given differential equation. (1)

15. If $y(x)$ is a solution of the differential equation

$$\left(\frac{2 + \sin x}{1 + y} \right) \frac{dy}{dx} = -\cos x \text{ and } y(0) = 1, \text{ then find}$$

the value of $y\left(\frac{\pi}{2}\right)$.

Delhi 2014C

Given differential equation is

$$\left(\frac{2 + \sin x}{1 + y} \right) \frac{dy}{dx} = -\cos x$$

$$\Rightarrow \frac{1}{1 + y} dy = -\frac{\cos x}{2 + \sin x} dx \quad (1)$$

Now, on integrating both sides, we get

$$\int \frac{1}{1 + y} dy = -\int \frac{\cos x}{2 + \sin x} dx$$

$$\Rightarrow \log|1 + y| = -\log|2 + \sin x| + \log C$$

$$\left[\begin{array}{l} \text{for } \int \frac{\cos x}{2 + \sin x} dx, \text{ let } 2 + \sin x = t \\ \qquad \qquad \qquad \Rightarrow \cos x dx = dt, \\ \text{then } \int \frac{\cos x}{2 + \sin x} dx = \int \frac{dt}{t} = \log t + C \\ \qquad \qquad \qquad = \log|2 + \sin x| + C \end{array} \right]$$

$$\Rightarrow \log(1 + y) + \log(2 + \sin x) = \log C$$

$$\Rightarrow \log(1 + y)(2 + \sin x) = \log C$$

$$\Rightarrow (1 + y)(2 + \sin x) = C \quad \dots(i)$$

Also, given that at $x = 0, y(0) = 1$

On putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$(1 + 1)(2 + \sin 0) = C$$

$$\Rightarrow C = 4 \quad (1)$$

On putting $C = 4$ in Eq. (i), we get

$$(1 + y)(2 + \sin x) = 4$$

$$\Rightarrow 1 + y = \frac{4}{2 + \sin x}$$

$$\Rightarrow y = \frac{4}{2 + \sin x} - 1$$

$$\Rightarrow y = \frac{4 - 2 - \sin x}{2 + \sin x}$$

$$\Rightarrow y = \frac{2 - \sin x}{2 + \sin x} \quad (1)$$

$$\text{Now, at } x = \frac{\pi}{2}, y\left(\frac{\pi}{2}\right) = \frac{2 - \sin \frac{\pi}{2}}{2 + \sin \frac{\pi}{2}}$$

$$\therefore y\left(\frac{\pi}{2}\right) = \frac{1}{3} \quad \left[\because \sin \frac{\pi}{2} = 1 \right] \quad (1)$$

16. Solve the differential equation

$$x \frac{dy}{dx} + y = x \cdot \cos x + \sin x, \text{ given } y\left(\frac{\pi}{2}\right) = 1.$$

All India 2014C

Given differential equation is

$$x \frac{dy}{dx} + y = x \cos x + \sin x$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$$

[dividing on both sides by x]

which is a linear differential equation.

On comparing with the form $\frac{dy}{dx} + Py = Q$,

we get $P = \frac{1}{x}$ and $Q = \cos x + \frac{\sin x}{x}$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

The general solution is given by

$$y \cdot \text{IF} = \int Q \times \text{IF} dx + C \quad (1)$$

$$\Rightarrow yx = \int x \left(\cos x + \frac{\sin x}{x} \right) dx + C$$

$$\Rightarrow yx = \int (x \cos x + \sin x) dx + C$$

$$\Rightarrow xy = \int x \cos x dx + \int \sin x dx + C$$

$$\Rightarrow xy = x \int \cos x dx - \int \left[\frac{d}{dx}(x) \int \cos x dx \right] dx + \int \sin x dx + C$$

$$\begin{aligned} \Rightarrow xy &= x \sin x + \cos x - \cos x + C \\ \Rightarrow xy &= x \sin x + C \\ \Rightarrow y &= \sin x + C \cdot \frac{1}{x} \quad \dots(i) \quad (1) \end{aligned}$$

Also, given that at $x = \frac{\pi}{2}; y = 1$

On putting $x = \frac{\pi}{2}$ and $y = 1$ in Eq. (i), we get

$$1 = 1 + C \cdot \frac{2}{\pi} \Rightarrow C = 0 \quad (1)$$

On putting the value of C in Eq. (i), we get

$$y = \sin x$$

which is the required solution of given differential equation. (1)

17. Solve the differential equation

$$\frac{dy}{dx} + y \cot x = 2 \cos x, \text{ given that } y = 0, \text{ when}$$

$$x = \frac{\pi}{2}.$$

Foreign 2014

Given differential equation is

$$\frac{dy}{dx} + y \cot x = 2 \cos x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = \cot x$ and $Q = 2 \cos x$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \cot x dx} = e^{\log \sin x}$$

$$\Rightarrow \text{IF} = \sin x \quad (1)$$

The general solution is given by

$$Y \times \text{IF} = \int \text{IF} \times Q dx + C$$

$$\Rightarrow y \sin x = \int 2 \sin x \cos x dx + C$$

$$\Rightarrow y \sin x = \int \sin 2x dx + C$$

$$\Rightarrow y \sin x = -\frac{\cos 2x}{2} + C \quad \dots(i) \quad (1)$$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

On putting $x = \frac{\pi}{2}$ and $y = 0$ in Eq. (i), we get

$$0 \sin \frac{\pi}{2} = -\frac{\cos 2 \frac{\pi}{2}}{2} + C$$

$$\Rightarrow C - \frac{\cos \pi}{2} = 0 \Rightarrow C + \frac{1}{2} = 0$$

$$\therefore C = -\frac{1}{2} \quad (1)$$

On putting the value of C in Eq. (i), we get

$$y \sin x = -\cos \frac{2x}{2} - \frac{1}{2}$$

$$\Rightarrow 2y \sin x + \cos 2x + 1 = 0$$

which is the required solution. (1)

- 18.** Solve the differential equation
 $(x^2 - yx^2) dy + (y^2 + x^2y^2) dx = 0$, given that
 $y = 1$, when $x = 1$. Foreign 2014

Direction (Q. Nos. 19-22) Solve the following differential equations.

Given differential equation is

$$(x^2 - yx^2)dy + (y^2 + x^2y^2) dx = 0$$

On dividing both sides by dx , we get

$$(x^2 - yx^2) \frac{dy}{dx} + (y^2 + x^2y^2) = 0$$

$$\Rightarrow x^2 (1 - y) \frac{dy}{dx} + y^2 (1 + x^2) = 0$$

$$\Rightarrow -x^2 (1 - y) \frac{dy}{dx} = y^2 (1 + x^2)$$

$$\Rightarrow x^2 (y-1) \frac{dy}{dx} = y^2 (1+x^2)$$

$$\Rightarrow \frac{y-1}{y^2} dy = \frac{1+x^2}{x^2} dx \quad (1)$$

On integrating both sides, we get

$$\int \frac{y-1}{y^2} dy = \int \frac{1+x^2}{x^2} dx$$

$$\Rightarrow \frac{1}{2} \int \frac{2y}{y^2} dy - \int \frac{1}{y^2} dy = \int \frac{1}{x^2} dx + \int 1 \cdot dx \quad (1)$$

On putting $y^2 = t \Rightarrow 2y dy = dt$ in first integral, we get

$$\frac{1}{2} \int \frac{dt}{t} + \frac{1}{y} = -\frac{1}{x} + x$$

$$\Rightarrow \frac{1}{2} \log |y^2| + \frac{1}{y} = -\frac{1}{x} + x + C \quad \dots(i)$$

$$[\because t = y^2]$$

Also, given that $y = 1$, when $x = 1$.

On putting $y = 1$ and $x = 1$ in Eq.(i), we get

$$\frac{1}{2} \log |1| + \frac{1}{1} = \frac{-1}{1} + 1 + C$$

$$\Rightarrow \frac{1}{2} \log |1| + 1 = -1 + 1 + C$$

$$\Rightarrow C = 1 \quad [\because \log 1 = 0] \quad (1)$$

On putting the value of C in Eq. (i), we get

$$\frac{1}{2} \log |y^2| + \frac{1}{y} = -\frac{1}{x} + x + 1$$

which is the required solution. (1)

19. $\frac{dy}{dx} + y \sec x = \tan x$ All India 2012C; Delhi 2008C

Given differential equation is

$$\frac{dy}{dx} + y \sec x = \tan x \quad \dots(i)$$

which is a linear differential equation of first order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \sec x \text{ and } Q = \tan x \quad (1)$$

$$\therefore \text{IF} = e^{\int \sec x \, dx} = e^{\log |\sec x + \tan x|}$$

$$[\because \int \sec x \, dx = \log |\sec x + \tan x|]$$

$$\Rightarrow \text{IF} = \sec x + \tan x \quad (1)$$

The general solution is

$$y \times \text{IF} = \int Q \cdot \text{IF} \, dx + C$$

$$y (\sec x + \tan x) = \int \tan x \cdot (\sec x + \tan x) \, dx$$

$$\Rightarrow y (\sec x + \tan x) = \int \sec x \tan x \, dx + \int \tan^2 x \, dx$$

$$\Rightarrow y (\sec x + \tan x) = \sec x + \int (\sec^2 x - 1) \, dx \quad (1)$$

$$\Rightarrow y (\sec x + \tan x) = (\sec x + \tan x) - x + C$$

$$[\because \int \sec^2 x \, dx = \tan x + C]$$

On dividing both sides by $(\sec x + \tan x)$, we get the required solution as

$$y = 1 - \frac{x}{\sec x + \tan x} + \frac{C}{\sec x + \tan x} \quad (1)$$

20. $2x^2 \frac{dy}{dx} - 2xy + y^2 = 0$

Delhi 2012

Given differential equation is

$$2x^2 \frac{dy}{dx} - 2xy + y^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \quad \dots(i) \quad (1)$$

which is a homogeneous differential equation.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$ in Eq. (i),

we get

$$v + x \frac{dv}{dx} = \frac{2vx^2 - v^2x^2}{2x^2} \quad (1)$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{2v - v^2}{2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v - v^2}{2} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v - v^2 - 2v}{2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-v^2}{2}$$

$$\Rightarrow \frac{2dv}{v^2} = -\frac{1}{x} dx \quad (1)$$

On integrating both sides, we get

$$\int \frac{2dv}{v^2} = \int \frac{-dx}{x} + C$$

$$\Rightarrow 2 \int v^{-2} dv = -\log|x| + C$$

$$\Rightarrow \frac{2v^{-1}}{-1} = -\log|x| + C$$

$$\Rightarrow \frac{-2}{v} = -\log|x| + C$$

$$\Rightarrow \frac{-2x}{y} = -\log|x| + C$$

$$\left[\because y = vx \Rightarrow v = \frac{y}{x} \right]$$

$$\Rightarrow -2x = y(-\log|x| + C)$$

$$\Rightarrow y = \frac{-2x}{-\log|x| + C}$$

which is the required solution.

(1)

21. $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$,
when $x = 0$.

Delhi 2012

Given differential equation is

$$\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2 \quad (1)$$

$$\Rightarrow \frac{dy}{dx} = (1 + x^2) + y^2(1 + x^2)$$

$$\Rightarrow \frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

$$\Rightarrow \frac{dy}{1 + y^2} = (1 + x^2) dx \quad (1)$$

On integrating both sides, we get

$$\int \frac{dy}{1 + y^2} = \int (1 + x^2) dx$$

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + C \quad \dots(i)$$

Also, given that $y = 0$, when $x = 2$.

On putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$\tan^{-1} 1 = C$$

$$\Rightarrow \tan^{-1}(\tan \pi/4) = C \quad \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\Rightarrow C = \pi/4 \quad (1)$$

On putting the value of C in Eq. (i), we get

$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}$$

$$\Rightarrow y = \tan \left(x + \frac{x^3}{3} + \frac{\pi}{4} \right)$$

which is the required solution. (1)

22. $x(x^2 - 1)\frac{dy}{dx} = 1$, $y = 0$, when $x = 2$.

All India 2012

Given differential equation is

$$x(x^2 - 1) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x(x^2 - 1)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x(x-1)(x+1)}$$

$$[\because a^2 - b^2 = (a-b)(a+b)]$$

$$\Rightarrow dy = \frac{dx}{x(x-1)(x+1)}$$

On integrating both sides, we get

$$\int dy = \int \frac{dx}{x(x-1)(x+1)} + C$$

$$\Rightarrow y = I + C \quad \dots(i)$$

$$\text{where, } I = \int \frac{dx}{x(x-1)(x+1)} \quad (1)$$

$$\text{Let } \frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\Rightarrow 1 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

On comparing coefficients of x^2 , x and constant terms from both sides, we get

$$A + B + C = 0 \quad \dots(ii)$$

$$B - C = 0 \quad \dots(iii)$$

$$\text{and } -A = 1$$

$$\Rightarrow A = -1$$

On putting $A = -1$ in Eq. (ii), we get

$$B + C = 1 \quad \dots(iv)$$

Now, on adding Eqs. (iii) and (iv), we get

$$2B = 1 \Rightarrow B = \frac{1}{2}$$

On putting $B = \frac{1}{2}$ in Eq. (iii), we get

$$\frac{1}{2} - C = 0 \Rightarrow C = \frac{1}{2}$$

$$\therefore A = -1, B = \frac{1}{2} \text{ and } C = \frac{1}{2},$$

$$\text{then } \frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1} \quad (1)$$

On integrating both sides w.r.t. x , we get

$$I = \int \frac{1}{x(x-1)(x+1)} dx = \int \frac{-1}{x} dx$$

$$+ \frac{1}{2} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1}$$

$$\Rightarrow I = -\log|x| + \frac{1}{2} \log|x-1| + \frac{1}{2} \log|x+1|$$

On putting the value of I in Eq. (i), we get

$$y = -\log|x| + \frac{1}{2} \log|x-1| + \frac{1}{2} \log|x+1| + C$$

... (v)

Also, given that $y = 0$, when $x = 2$.

On putting $y = 0$ and $x = 2$ in Eq. (v), we get

$$0 = -\log 2 + \frac{1}{2} \log 1 + \frac{1}{2} \log 3 + C$$

$$\Rightarrow C = \log 2 - \frac{1}{2} \log 1 - \frac{1}{2} \log 3$$

$$\Rightarrow C = \log 2 - \log \sqrt{3} \quad [:\because \log 1 = 0]$$

$$\Rightarrow C = \log \frac{2}{\sqrt{3}} \quad (1)$$

On putting the value of C in Eq. (v), we get

$$y = -\log|x| + \frac{1}{2} \log|x-1|$$

$$+ \frac{1}{2} \log|x+1| + \log \frac{2}{\sqrt{3}} \quad (1)$$

which is the required solution.

23. Solve the following differential equation

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x, \text{ given that } y = 0,$$

$$\text{when } x = \frac{\pi}{2}.$$

Delhi 2012C; Foreign 2011

Given differential equation is

$$\frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$$

which is a linear differential equation.

On comparing with general form of linear differential equation of 1st order

$$\frac{dy}{dx} + Py = Q, \text{ we get}$$

$$P = \cot x \text{ and } Q = 4x \operatorname{cosec} x \quad (1)$$

$$\begin{aligned} \therefore \text{IF} &= e^{\int P dx} = e^{\int \cot x dx} \\ &= e^{\log \sin x} = \sin x \quad [\because e^{\log x} = x] \end{aligned}$$

$$\Rightarrow \text{IF} = \sin x \quad (1)$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

On putting $\text{IF} = \sin x$ and $Q = 4x \operatorname{cosec} x$, we get

$$y \times \sin x = \int 4x \operatorname{cosec} x \cdot \sin x dx + C$$

$$\Rightarrow y \sin x = \int 4x \cdot \frac{1}{\sin x} \cdot \sin x dx + C$$

$$\Rightarrow y \sin x = \int 4x dx + C$$

$$\Rightarrow y \sin x = 2x^2 + C \quad \dots(i) \quad (1)$$

Also, given that $y = 0$, when $x = \frac{\pi}{2}$.

On putting $y = 0$ and $x = \frac{\pi}{2}$ in Eq. (i), we get

$$0 = 2 \times \frac{\pi^2}{4} + C \Rightarrow C = \frac{-\pi^2}{2}$$

On putting $C = -\frac{\pi^2}{2}$ in Eq. (i), we get

$$y \sin x = 2x^2 - \frac{\pi^2}{2}$$

$$\Rightarrow y = 2x^2 \operatorname{cosec} x - \frac{\pi^2}{2} \operatorname{cosec} x \quad (1)$$

which is the required solution.

24. Solve the following differential equation
 $(1+x^2) dy + 2xy dx = \cot x dx$, where $x \neq 0$.

All India 2012C, 2011

Given differential equation is

$$(1+x^2) dy + 2xy dx = \cot x dx \quad [\because x \neq 0]$$

$$\Rightarrow (1+x^2) dy = (\cot x - 2xy) dx$$

On dividing both sides by $1+x^2$, we get

$$dy = \frac{\cot x - 2xy}{1+x^2} dx$$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{\cot x}{1+x^2} \quad \dots(i) \quad (1)$$

which is a linear differential equation of 1st order and is of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{1+x^2} \quad \text{and} \quad Q = \frac{\cot x}{1+x^2}$$

$$\therefore \text{IF} = e^{\int \frac{2x}{1+x^2} dx}$$

$$= e^{\log|1+x^2|} = 1+x^2 \quad (1)$$

$$\left[\text{for } \int \frac{2x}{1+x^2} dx, \text{ put } 1+x^2 = t \Rightarrow 2x dx = dt \right]$$

$$\left[\int \frac{dt}{t} = \log |t| = \log |1 + x^2| + C \right]$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore y(1 + x^2) = \int \frac{\cot x}{1 + x^2} \times (1 + x^2) dx + C$$

$$\Rightarrow y(1 + x^2) = \int \cot x dx + C \quad (1)$$

$$\Rightarrow y(1 + x^2) = \log |\sin x| + C$$

$$[\because \int \cot x dx = \log |\sin x| + C]$$

On dividing both sides by $1 + x^2$, we get

$$y = \frac{\log |\sin x|}{1 + x^2} + \frac{C}{1 + x^2}$$

which is the required solution. (1)

25. Find the particular solution of the differential equation

$$(1 + e^{2x})dy + (1 + y^2)e^x dx = 0, \text{ given that } y = 1, \\ \text{when } x = 0. \quad \text{Foreign 2011; All India 2008C}$$

Given differential equation is

$$(1 + e^{2x}) dy + (1 + y^2)e^x dx = 0$$

Above equation may be written as

$$\frac{dy}{1 + y^2} = \frac{-e^x}{1 + e^{2x}} dx \quad (1)$$

On integrating both sides, we get

$$\int \frac{dx}{1+y^2} = -\int \frac{e^x}{1+e^{2x}} dx$$

On putting $e^x = t \Rightarrow e^x dx = dt$ in RHS, we get

$$\tan^{-1} y = -\int \frac{1}{1+t^2} dt$$

$$\Rightarrow \tan^{-1} y = -\tan^{-1} t + C$$

$$\Rightarrow \tan^{-1} y = -\tan^{-1}(e^x) + C \quad \dots(i)$$

$$[\because t = e^x](1\frac{1}{2})$$

Now, given that $y = 1$, when $x = 0$.

On putting above values in Eq. (i), we get

$$\tan^{-1} 1 = -\tan^{-1}(e^0) + C$$

$$\Rightarrow \tan^{-1}\left(\tan \frac{\pi}{4}\right) = -\tan^{-1} 1 + C \quad [\because e^0 = 1]$$

$$\Rightarrow \frac{\pi}{4} = -\tan^{-1}\left(\tan \frac{\pi}{4}\right) + C$$

$$\Rightarrow \frac{\pi}{4} = -\frac{\pi}{4} + C$$

$$\Rightarrow C = \frac{\pi}{4} + \frac{\pi}{4} \Rightarrow C = \frac{\pi}{2}$$

On putting $C = \frac{\pi}{2}$ in Eq. (i), we get

$$\tan^{-1} y = -\tan^{-1} e^x + \frac{\pi}{2}$$

$$\Rightarrow y = \tan \left[\frac{\pi}{2} - \tan^{-1}(e^x) \right] = \cot [\tan^{-1}(e^x)]$$

$$= \cot \left[\cot^{-1} \left(\frac{1}{e^x} \right) \right] \quad \left[\because \tan^{-1} x = \cot^{-1} \frac{1}{x} \right]$$

$$\Rightarrow y = \frac{1}{e^x}$$

which is the required solution.

(1½)

26. Solve the following differential equation

$$(1 + x^2) \frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}, \text{ given that } y = 0,$$

when $x = 1$.

Foreign 2011

Given differential equation is

$$(1 + x^2) \frac{dy}{dx} + 2xy = \frac{1}{1 + x^2}$$

On dividing both sides by $(1 + x^2)$, we get

$$\frac{dy}{dx} + \frac{2xy}{1 + x^2} = \frac{1}{(1 + x^2)^2} \quad \dots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{1}{(1+x^2)^2} \quad (1)$$

$$\therefore \text{IF} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log|1+x^2|} \quad (1)$$

$$\Rightarrow \text{IF} = 1+x^2 \quad [\because e^{\log x} = x]$$

$$\left[\begin{array}{l} \because \int \frac{2x}{1+x^2} dx, \text{ put } 1+x^2 = t \Rightarrow 2x dx = dt \\ \therefore \int \frac{dt}{t} = \log|t| = \log|1+x^2| \end{array} \right]$$

Now, solution of linear equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C \quad \dots(\text{iii})$$

$$\therefore y(1+x^2) = \int \frac{1}{(1+x^2)^2} \times (1+x^2) dx + C$$

$$\Rightarrow y(1+x^2) = \int \frac{1}{1+x^2} dx + C$$

$$\Rightarrow y(1+x^2) = \tan^{-1} x + C \quad \dots(\text{iv}) \quad (1)$$

$$\left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x + C \right]$$

Also, given that $y = 0$, when $x = 1$.

On putting $y = 0$ and $x = 1$ in Eq. (iv), we get

$$0 = \tan^{-1} 1 + C$$

$$\Rightarrow 0 = \tan^{-1} \left(\tan \frac{\pi}{4} \right) + C \quad \left[\because 1 = \tan \frac{\pi}{4} \right]$$

$$\Rightarrow 0 = \frac{\pi}{4} + C \Rightarrow C = -\frac{\pi}{4}$$

On putting $C = -\frac{\pi}{4}$ in Eq. (iv), we get

$$y(1+x^2) = \tan^{-1} x - \frac{\pi}{4}$$

$$\Rightarrow y = \frac{\tan^{-1} x}{1+x^2} - \frac{\pi}{4(1+x^2)} \quad (1)$$

which is the required solution.

27. Solve the following differential equation

$$x dy - y dx = \sqrt{x^2 + y^2} dx.$$

All India 2011

Given differential equation is

$$x dy - y dx = \sqrt{x^2 + y^2} dx$$

$$\Rightarrow (y + \sqrt{x^2 + y^2}) dx = x dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x} \quad \dots(i) \quad (1)$$

which is a homogeneous differential equation because each term have same degree.

$$\text{On putting } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2 x^2}}{x} = \frac{vx + x\sqrt{1 + v^2}}{x}$$

$$\Rightarrow v + x \frac{dv}{dx} = v + \sqrt{1 + v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \sqrt{1 + v^2} \Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \log |v + \sqrt{1 + v^2}| = \log |x| + C$$

$$\left[\because \int \frac{dx}{\sqrt{a^2 + x^2}} = \log |x + \sqrt{x^2 + a^2}| \right.$$

$$\left. \text{and } \int \frac{dx}{x} = \log |x| + C \right] \quad (1)$$

$$\Rightarrow \log \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = \log |x| + C \quad \left[\because y = vx \right]$$

$$\left| x \quad \vee \quad x^2 \right| \quad \left[\because v = \frac{y}{x} \right]$$

$$\Rightarrow \log \left| \frac{y + \sqrt{x^2 + y^2}}{x} \right| - \log |x| = C$$

$$\Rightarrow \log \frac{y + \sqrt{x^2 + y^2}}{x} = C$$

$$\left[\because \log m - \log n = \log \left(\frac{m}{n} \right) \right]$$

$$\Rightarrow \frac{y + \sqrt{x^2 + y^2}}{x^2} = e^C \quad \left[\because \text{if } \log y = x, \right. \\ \left. \text{then } y = e^x \right]$$

$$\Rightarrow y + \sqrt{x^2 + y^2} = x^2 \cdot e^C$$

$$\therefore y + \sqrt{x^2 + y^2} = Ax^2 \quad \left[\text{where, } A = e^C \right] \text{(1)}$$

which is the required solution.

28. Solve the following differential equation

$$(y + 3x^2) \frac{dx}{dy} = x.$$

All India 2011

Given differential equation is

$$(y + 3x^2) \frac{dx}{dy} = x \Rightarrow \frac{dy}{dx} = \frac{y}{x} + 3x$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 3x \quad \dots \text{(i) (1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{-1}{x} \text{ and } Q = 3x \quad (1)$$

$$\therefore \text{IF} = e^{\int \frac{-1}{x} dx} = e^{-\log|x|} = e^{\log x^{-1}} = x^{-1}$$

$$\Rightarrow \text{IF} = x^{-1} = \frac{1}{x}$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore y \times \frac{1}{x} = \int 3x \times \frac{1}{x} dx \quad (1)$$

$$\Rightarrow \frac{y}{x} = \int 3 dx \Rightarrow \frac{y}{x} = 3x + C$$

$$\Rightarrow y = 3x^2 + Cx$$

which is the required solution. (1)

29. Solve the following differential equation

$$x dy - (y + 2x^2) dx = 0.$$

All India 2011

Given differential equation is

$$x dy - (y + 2x^2) dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y + 2x^2}{x}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x \quad \dots(i) \quad (1)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{-1}{x} \text{ and } Q = 2x \quad (1)$$

$$\therefore \text{IF} = e^{\int \frac{-1}{x} dx} = e^{-\log|x|} = x^{-1} = \frac{1}{x} \quad (1)$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore \frac{y}{x} = \int (2x \times \frac{1}{x}) dx + C$$

$$\Rightarrow \frac{y}{x} = \int 2 dx + C \Rightarrow \frac{y}{x} = 2x + C$$

$$\Rightarrow y = 2x^2 + Cx$$

which is the required solution. (1)

30. Solve the following differential equation

$$x dy + (y - x^3) dx = 0.$$

All India 2011

Given differential equation is

$$x dy + (y - x^3) dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 - y}{x}$$

$$\Rightarrow \frac{dy}{dx} = x^2 - \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2 \quad \dots(i) \quad (1)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x} \text{ and } Q = x^2 \quad (1)$$

$$\therefore \text{IF} = e^{\int \frac{1}{x} dx} = e^{\log|x|} = x \quad (1)$$

Solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore y \times x = \int x^2 \times x dx + C$$

$$\Rightarrow yx = \int x^3 dx + C$$

$$\Rightarrow yx = \frac{x^4}{4} + C \Rightarrow y = \frac{x^3}{4} + \frac{C}{x}$$

which is the required solution. (1)

31. Solve the following differential equation

$$e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0. \quad \text{Delhi 2011}$$

Given differential equation is

$$e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$$

$$\Rightarrow \frac{e^x}{e^x - 1} dx = \frac{\sec^2 y}{\tan y} dy \quad (1)$$

On integrating both sides, we get

$$\int \frac{e^x}{e^x - 1} dx = \int \frac{\sec^2 y}{\tan y} dy$$

On putting $e^x - 1 = t$ and $\tan y = z$

$$\Rightarrow e^x dx = dt \text{ and } \sec^2 y \, dy = dz$$

$$\therefore \int \frac{dt}{t} = \int \frac{dz}{z} \quad (1)$$

$$\Rightarrow \log |t| = \log |z| + \log C \left[\because \int \frac{1}{x} dx = \log |x| \right]$$

$$\Rightarrow \log |e^x - 1| = \log |\tan y| + \log C$$

$$\Rightarrow \log |e^x - 1| = \log |C \cdot \tan y|$$

$$\left[\because \log m + \log n = \log mn \right]$$

$$\Rightarrow e^x - 1 = C \tan y \quad (1)$$

$$\Rightarrow \tan y = \frac{e^x - 1}{C} \Rightarrow y = \tan^{-1} \left(\frac{e^x - 1}{C} \right)$$

which is the required solution. (1)

32. Solve the following differential equation

$$(1 + y^2)(1 + \log x) dx + x dy = 0. \quad \text{Delhi 2011}$$

Given differential equation is

$$(1 + y^2)(1 + \log x) dx + x dy = 0$$

$$\Rightarrow \frac{1 + \log x}{x} dx = \frac{-dy}{1 + y^2} \quad (1)$$

On integrating both sides, we get

$$\int \frac{1 + \log x}{x} dx = - \int \frac{dy}{1 + y^2}$$

$$\Rightarrow \int \frac{1}{x} dx + \int \frac{\log x}{x} dx = - \int \frac{dy}{1 + y^2} \quad (1\frac{1}{2})$$

$$\Rightarrow \log|x| + \frac{(\log x)^2}{2} + C = - \tan^{-1} y$$

$$\left[\begin{array}{l} \text{for } \int \frac{\log x}{x} dx \Rightarrow \text{put } \log x = t \Rightarrow \frac{1}{x} dx = dt \\ \therefore \int t dt = \frac{t^2}{2} + C = \frac{(\log x)^2}{2} + C \end{array} \right]$$

$$\Rightarrow \tan^{-1} y = - \left[\log|x| + \frac{(\log x)^2}{2} + C \right]$$

$$\Rightarrow y = \tan \left[- \log|x| - \frac{(\log x)^2}{2} - C \right]$$

which is the required solution. (1\frac{1}{2})

33. Solve the following differential equation

$$\left[x \sin^2 \left(\frac{y}{x} \right) - y \right] dx + x dy = 0. \quad \text{Delhi 2011C}$$

Given differential equation is

$$\left[x \sin^2\left(\frac{y}{x}\right) - y \right] dx + x dy = 0$$

which is a homogeneous differential equation.

This equation can be written as

$$\left[x \sin^2\left(\frac{y}{x}\right) - y \right] dx = -x dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \quad \dots(i)$$

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$ in

Eq. (i), we get **(1)**

$$v + x \frac{dv}{dx} = \frac{vx - x \sin^2\left(\frac{vx}{x}\right)}{x} = v - \sin^2 v$$

$$\Rightarrow x \frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \frac{dv}{\sin^2 v} = -\frac{dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{dv}{\sin^2 v} = -\int \frac{dx}{x}$$

$$\Rightarrow \int \operatorname{cosec}^2 v \, dv = -\int \frac{dx}{x} \left[\because \frac{1}{\sin^2 v} = \operatorname{cosec}^2 v \right]$$

$$\Rightarrow -\cot v = -\log x + C$$

$$\left[\because \int \operatorname{cosec}^2 v \, dv = -\cot v + C \right] (1)$$

$$\Rightarrow -\cot \left(\frac{y}{x} \right) = -\log x + C \left[\because y = vx \therefore v = \frac{y}{x} \right]$$

$$\Rightarrow \cot \left(\frac{y}{x} \right) = \log x - C$$

$$\Rightarrow \frac{y}{x} = \cot^{-1}(\log x - C) \quad (1)$$

$$\Rightarrow y = x \cdot \cot^{-1}(\log x - C)$$

which is the required solution.

34. Solve the following differential equation

$$x \frac{dy}{dx} + y - x + xy \cot x = 0, \quad x \neq 0. \quad \text{Delhi 2011C}$$

Given differential equation is

$$x \frac{dy}{dx} + y - x + xy \cot x = 0, x \neq 0$$

Above equation can be written as

$$x \frac{dy}{dx} + y(1 + x \cot x) = x$$

On dividing both sides by x , we get

$$\frac{dy}{dx} + y \left(\frac{1 + x \cot x}{x} \right) = 1$$

$$\Rightarrow \frac{dy}{dx} + y \left(\frac{1}{x} + \cot x \right) = 1 \quad \dots(i) \quad \mathbf{(1)}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x} + \cot x \text{ and } Q = 1$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \left(\frac{1}{x} + \cot x \right) dx} = e^{\log|x| + \log \sin x}$$

$$\left[\because \int \frac{1}{x} dx = \log|x| \text{ and } \int \cot x dx = \log|\sin x| \right]$$

$$= e^{\log|x \sin x|}$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \text{IF} = x \sin x \quad (1/2)$$

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C \quad (1/2)$$

$$y \times x \sin x = \int 1 \times x \sin x dx + C$$

$$\Rightarrow y \times x \sin x = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin x} dx + C$$

$$\Rightarrow y \times x \sin x = x \int \sin x dx - \int \left(\frac{d}{dx}(x) \cdot \int \sin x dx \right) dx + C$$

$$[\text{using integration by parts in } \int x \sin x dx]$$

$$\Rightarrow y \times x \sin x = -x \cos x - \int 1(-\cos x) dx + C \quad (1)$$

$$\Rightarrow y \times x \sin x = -x \cos x + \int \cos x dx + C$$

$$\Rightarrow y \times x \sin x = -x \cos x + \sin x + C$$

On dividing both sides by $x \sin x$, we get

$$y = \frac{-x \cos x + \sin x + C}{x \sin x}$$

$$\Rightarrow y = -\cot x + \frac{1}{x} + \frac{C}{x \sin x}$$

which is the required solution. (1)

35. Show that the following differential equation is homogeneous and then solve it.

$$y dx + x \log \left(\frac{y}{x} \right) dy - 2x dy = 0$$

HOTS; All India 2011C



Let the value of $\frac{dy}{dx}$ be $f(x, y)$. Now, put $x = \lambda x$

and $y = \lambda y$ and verify whether $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ $n \in \mathbb{Z}$. If above equation is satisfied, then given equation is said to be homogeneous equation. Then, we use the substitution $y = vx$ to solve the equation.

Given differential equation is

$$y dx + x \log \left(\frac{y}{x} \right) dy - 2x dy = 0$$

$$\Rightarrow y dx = \left[2x - x \log \left(\frac{y}{x} \right) \right] dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x \log \left(\frac{y}{x} \right)} \quad \dots(i) \quad (1/2)$$

$$\text{Now, let } f(x, y) = \frac{y}{2x - x \log \left(\frac{y}{x} \right)}$$

On replace x by λx and y by λy both sides, we get

$$\begin{aligned} f(\lambda x, \lambda y) &= \frac{\lambda y}{2\lambda x - \lambda x \log \left(\frac{\lambda y}{\lambda x} \right)} \\ &= \frac{\lambda y}{\lambda \left[2x - x \log \left(\frac{y}{x} \right) \right]} \end{aligned}$$

$$\Rightarrow f(\lambda x, \lambda y) = \lambda^0 \frac{y}{2x - x \log \left(\frac{y}{x} \right)} = \lambda^0 f(x, y)$$

So, given differential equation is homogeneous.

(1/2)

$dv \quad dv$

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{vx}{2x - x \log \left(\frac{vx}{x} \right)} = \frac{v}{2 - \log v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{2 - \log v} - v = \frac{v - 2v + v \log v}{2 - \log v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-v + v \log v}{2 - \log v}$$

$$\Rightarrow \frac{2 - \log v}{v \log v - v} dv = \frac{dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{2 - \log v}{v(\log v - 1)} dv = \int \frac{dx}{x}$$

On putting $\log v = t \Rightarrow \frac{1}{v} dv = dt$

Then, $\int \frac{2-t}{t-1} dt = \log |x| + C$

$$\Rightarrow \int \left(\frac{1}{t-1} - 1 \right) dt = \log |x| + C \quad (1)$$

$$\left[\begin{array}{l} \because t-1 \overline{) 2-t} \quad (-1) \\ \underline{1-t} \\ \quad -+ \\ \quad \quad 1 \\ \text{and use } \int \left(\frac{R}{D} + Q \right) dt \end{array} \right]$$

$$\Rightarrow \log |t-1| - t = \log |x| + C$$

$$\Rightarrow \log |\log v - 1| - \log v = \log |x| + C$$

$$\Rightarrow \log \left| \frac{\log v - 1}{v} \right| = \log |x| + C$$

$$\left[\because \log m - \log n = \log \left(\frac{m}{n} \right) \right]$$

$$\Rightarrow \log \left| \frac{\log v - 1}{v} \right| - \log |x| = C$$

$$\Rightarrow \log \left| \frac{\log v - 1}{vx} \right| = C$$

$$\therefore \log \left| \frac{\log \frac{y}{x} - 1}{\frac{y}{x}} \right| = C \quad \left[\because y = vx \Rightarrow v = \frac{y}{x} \right]$$

which is the required solution. (1)

36. Solve the following differential equation

$$\left(x \cos \frac{y}{x} + y \sin \frac{y}{x} \right) y - \left(y \sin \frac{y}{x} - x \cos \frac{y}{x} \right) x \frac{dy}{dx} = 0.$$

All India 2010C



Firstly, convert the given differential equation in homogeneous and then put $y = vx$.

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Further, separate the variables and integrate it.

Given differential equation is

$$\left(x \cos \frac{y}{x} + y \sin \frac{y}{x} \right) \cdot y$$

$$- \left(y \sin \frac{y}{x} - x \cos \frac{y}{x} \right) \cdot x \frac{dy}{dx} = 0$$

which is a homogeneous differential equation.

It can be written as

$$\left(x \cos \frac{y}{x} + y \sin \frac{y}{x} \right) \cdot y$$

$$= \left(y \sin \frac{y}{x} - x \cos \frac{y}{x} \right) \cdot x \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[x \cos \left(\frac{y}{x} \right) + y \sin \left(\frac{y}{x} \right) \right] \cdot y}{\left(y \sin \frac{y}{x} - x \cos \frac{y}{x} \right) \cdot x} \quad \dots(i)$$

On putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \text{ in Eq. (i), we get } \quad (1)$$

$$v + x \frac{dv}{dx} = \frac{(x \cos v + vx \sin v) \cdot vx}{(vx \sin v - x \cos v) \cdot x}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v \cos v + v^2 \sin v}{v \sin v - \cos v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + v^2 \sin v}{v \sin v - \cos v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \cos v + v^2 \sin v - v^2 \sin v + v \cos v}{v \sin v - \cos v}$$

(1)

$$\Rightarrow x \frac{dv}{dx} = \frac{2v \cos v}{v \sin v - \cos v}$$
$$\Rightarrow \frac{v \sin v - \cos v}{v \cos v} dv = 2 \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{v \sin v - \cos v}{v \cos v} dv = 2 \int \frac{dx}{x}$$
$$\Rightarrow \int \left(\frac{v \sin v}{v \cos v} - \frac{\cos v}{v \cos v} \right) dv = 2 \int \frac{dx}{x}$$
$$\Rightarrow \int \left(\tan v - \frac{1}{v} \right) dv = 2 \int \frac{dx}{x}$$
$$\Rightarrow \log |\sec v| - \log |v| = 2 \log |x| + C \quad (1)$$
$$\left[\because \int \tan v dv = \log |\sec v| \text{ and } \int \frac{1}{x} dx = \log |x| \right]$$
$$\Rightarrow \log |\sec v| - \log |v| - 2 \log |x| = C$$
$$\Rightarrow \log |\sec v| - [\log |v| + \log |x|^2] = C$$

$$[\because \log m^n = n \log m]$$

$$\Rightarrow \log |\sec v| - \log |vx^2| = C$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \log \left| \frac{\sec v}{vx^2} \right| = C$$

$$\left[\because \log m - \log n = \log \left(\frac{m}{n} \right) \right]$$

$$\Rightarrow \log \left| \frac{\sec \frac{y}{x}}{\frac{y}{x} \cdot x^2} \right| = C \quad \left[\because y = vx \right]$$
$$\left[\because v = \frac{y}{x} \right]$$

$$\Rightarrow \log \left| \frac{\sec \frac{y}{x}}{xy} \right| = C$$

which is the required solution

(1)

which is the required solution.

37. Solve the following differential equation

$$xy \log \left(\frac{y}{x} \right) dx + \left[y^2 - x^2 \log \left(\frac{y}{x} \right) \right] dy = 0. \quad \text{Delhi 2010C}$$

Given differential equation is

$$xy \log \left(\frac{y}{x} \right) dx + \left[y^2 - x^2 \log \left(\frac{y}{x} \right) \right] dy = 0$$

which is a homogeneous differential equation. This equation can be written as

$$xy \log \left(\frac{y}{x} \right) dx = \left[x^2 \log \left(\frac{y}{x} \right) - y^2 \right] dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy \log \left(\frac{y}{x} \right)}{x^2 \log \left(\frac{y}{x} \right) - y^2} \quad \dots(i)$$

$$\text{Now, put } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{vx^2 \log \left(\frac{vx}{x} \right)}{x^2 \log \left(\frac{vx}{x} \right) - v^2 x^2} = \frac{v \log v}{\log v - v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \log v}{\log v - v^2} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v \log v - v \log v + v^3}{\log v - v^2} = \frac{v^3}{\log v - v^2}$$

$$\Rightarrow \frac{\log v - v^2}{v^3} dv = \frac{dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{\log v - v^2}{v^3} dv = \int \frac{dx}{x}$$

$\int \frac{\log v}{v^3} dv - \int \frac{v^2}{v^3} dv = \int \frac{dx}{x}$

$$\Rightarrow \int \frac{v^{-3}}{v^3} dv - \int \frac{1}{v} dv = \int \frac{1}{x}$$

$$\Rightarrow \int v^{-3} \log v dv - \log |v| = \log |x| + C$$

Using integration by parts, we get

$$\log v \int v^{-3} dv - \int \left[\frac{d}{dv} (\log v) \cdot \int v^{-3} dv \right] dv$$

$$= \log |v| + \log |x| + C$$

$$\Rightarrow \frac{v^{-2}}{-2} \log v - \int \frac{1}{v} \frac{v^{-2}}{-2} dv = \log |v| + \log |x| + C$$

$$\Rightarrow \frac{-1}{2v^2} \log v + \frac{1}{2} \int v^{-3} dv = \log |v| + \log |x| + C$$

$$\Rightarrow \frac{-1}{2v^2} \log v + \frac{1}{2} \cdot \frac{v^{-2}}{-2} = \log |v| + \log |x| + C$$

$$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + C \right]$$

$$\Rightarrow \frac{-1}{2v^2} \log v - \frac{1}{4v^2} = \log |vx| + C \quad (1)$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \frac{-1}{2} \cdot \frac{x^2}{y^2} \log \left(\frac{y}{x} \right) - \frac{1}{4} \cdot \frac{x^2}{y^2} = \log \left| \frac{y}{x} \cdot x \right| + C$$

$$\left[\because y = vx \Rightarrow v = \frac{y}{x} \right]$$

$$\Rightarrow \frac{-x^2}{2y^2} \log \left(\frac{y}{x} \right) - \frac{x^2}{4y^2} = \log |y| + C$$

$$\Rightarrow \frac{-x^2}{y^2} \left[\frac{\log \left(\frac{y}{x} \right)}{2} + \frac{1}{4} \right] = \log |y| + C$$

$$\Rightarrow \frac{x^2}{4y^2} \left[2 \log \left(\frac{y}{x} \right) + 1 \right] + \log |y| = -C$$

$$\Rightarrow x^2 \left[2 \log \left(\frac{y}{x} \right) + 1 \right] + 4y^2 \log |y| = 4y^2 k$$

[where, $k = -C$] (1)

which is the required solution.

38. Solve the following differential equation

$$(x^2 + 1) \frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}. \text{ All India 2010, 2008}$$

Given differential equation is

$$(x^2 + 1) \frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$$

On dividing both sides by $(x^2 + 1)$, we get

$$\frac{dy}{dx} + \frac{2xy}{x^2 + 1} = \frac{\sqrt{x^2 + 4}}{x^2 + 1} \quad \dots(i)$$

which is a linear differential equation of the

form $\frac{dy}{dx} + Py = Q \quad \dots(ii)$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{2x}{x^2 + 1} \quad \text{and} \quad Q = \frac{\sqrt{x^2 + 4}}{x^2 + 1}$$

$$\therefore \text{IF} = e^{\int \frac{2x}{x^2 + 1} dx} = e^{\log|x^2 + 1|}$$

$$\Rightarrow \text{IF} = x^2 + 1 \quad [\because e^{\log x} = x] \quad (1)$$

$$\left[\because \int \frac{2x}{x^2 + 1} dx \Rightarrow \text{put } x^2 + 1 = t \Rightarrow 2x dx = dt \right.$$

$$\left. \therefore \int \frac{dt}{t} = \log|t| = \log|x^2 + 1| \right]$$

Now, solution of this equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C \quad (1)$$

$$\therefore y(x^2 + 1) = \int (x^2 + 1) \cdot \frac{\sqrt{x^2 + 4}}{x^2 + 1} dx \quad (1)$$

$$\Rightarrow y(x^2 + 1) = \int \sqrt{x^2 + 4} dx$$

$$\Rightarrow y(x^2 + 1) = \int \sqrt{x^2 + (2)^2} dx$$

Now, we know that

$$\int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$$

$$\therefore y(x^2 + 1) = \frac{x}{2} \sqrt{x^2 + 4} + \frac{4}{2} \log |x + \sqrt{x^2 + 4}| + C$$

$$\Rightarrow y(x^2 + 1) = \frac{x}{2} \sqrt{x^2 + 4} + 2 \log |x + \sqrt{x^2 + 4}| + C$$

which is the required solution. (1)

39. Solve the following differential equation

$$(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x.$$

HOTS; All India 2010



Firstly, divide given equation by $x^3 + x^2 + x + 1$, then it becomes a variable separable type differential equation and then solve it.

Given differential equation is

$$(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x^2 + x}{x^3 + x^2 + x + 1}$$

It is a variable separable type differential equation.

$$dy = \frac{2x^2 + x}{x^3 + x^2 + x + 1} dx$$

On integrating both sides, we get

$$\int dy = \int \frac{2x^2 + x}{x^3 + x^2 + x + 1} dx$$

$$\Rightarrow y = \int \frac{2x^2 + x}{x^2(x+1) + 1(x+1)} dx$$

$$= \int \frac{2x^2 + x}{(x+1)(x^2+1)} dx$$

$$y = I \quad \dots(i) \quad (1)$$

where, $I = \int \frac{2x^2 + x}{(x+1)(x^2+1)} dx$

Using partial fractions, we get

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \quad \dots(ii)$$

$$\Rightarrow \frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$\Rightarrow 2x^2 + x = A(x^2+1) + (Bx+C)(x+1)$$

Now, comparing coefficients of x^2 , x and constant term from both sides, we get

constant term from both sides, we get

$$A + B = 2 \quad \dots(\text{iii})$$

$$B + C = 1 \quad \dots(\text{iv})$$

and $A + C = 0 \quad \dots(\text{v})$

On subtracting Eq. (iv) from Eq. (iii), we get

$$A - C = 1 \quad \dots(\text{vi})$$

On adding Eqs. (v) and (vi), we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

On putting $A = \frac{1}{2}$ in Eq. (iii), we get

$$\frac{1}{2} + B = 2 \Rightarrow B = 2 - \frac{1}{2} = \frac{3}{2}$$

On putting $B = \frac{3}{2}$ in Eq. (iv), we get

$$\frac{3}{2} + C = 1 \Rightarrow C = 1 - \frac{3}{2}$$

$$\Rightarrow C = \frac{-1}{2} \quad \text{(1)}$$

On substituting the values of A , B and C in Eq. (ii), we get

$$\frac{2x^2 + x}{(x+1)(x^2+1)} = \frac{1/2}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$

On integrating both sides, we get

$$I = \int \frac{2x^2 + x}{(x+1)(x^2+1)} dx = \frac{1}{2} \int \frac{dx}{x+1} + \frac{3}{2} \int \frac{x}{x^2+1} dx - \frac{1}{2} \int \frac{dx}{x^2+1}$$

$$\Rightarrow I = \frac{1}{2} \log|x+1| + \frac{3}{4} \log|x^2+1| - \frac{1}{2} \tan^{-1} x + C \quad \text{(1)}$$

$$\left[\because \int \frac{x}{x^2+1} dx \Rightarrow \text{put } x^2+1=t \Rightarrow 2x dx = dt \right]$$

$$\Rightarrow x dx = \frac{dt}{2}, \text{ then } \int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \log t = \frac{1}{2} \log |x^2+1| + C$$

On putting above value of I in Eq. (i), we get

$$y = \frac{1}{2} \log |x+1| + \frac{3}{4} \log |x^2+1| - \frac{1}{2} \tan^{-1} x + C$$

which is the required solution. **(1)**

40. Solve the following differential equation

$$\sqrt{1+x^2+y^2+x^2y^2} + xy \frac{dy}{dx} = 0. \text{ All India 2010}$$

Given differential equation is

$$\sqrt{1+x^2+y^2+x^2y^2} + xy \frac{dy}{dx} = 0$$

$$\Rightarrow \sqrt{(1+x^2)+y^2(1+x^2)} = -xy \frac{dy}{dx}$$

$$\Rightarrow \sqrt{(1+x^2)(1+y^2)} = -xy \frac{dy}{dx}$$

$$\Rightarrow \sqrt{1+x^2} \cdot \sqrt{1+y^2} = -xy \frac{dy}{dx}$$

$$\Rightarrow \frac{y \cdot dy}{\sqrt{1+y^2}} = -\frac{\sqrt{1+x^2}}{x} dx \quad (1)$$

On integrating both sides, we get

$$\int \frac{y}{\sqrt{1+y^2}} dy = - \int \frac{\sqrt{1+x^2}}{x^2} \cdot x dx$$

On putting $1+y^2 = t$ and $1+x^2 = u^2$

$$\Rightarrow 2y dy = dt \text{ and } 2x dx = 2u du$$

$$\Rightarrow y dy = \frac{dt}{2} \text{ and } x dx = u du \quad (1)$$

$$\therefore \frac{1}{2} \int \frac{dt}{\sqrt{t}} = - \int \frac{u}{u^2-1} \cdot u du$$

$$\Rightarrow \frac{1}{2} \int t^{-1/2} dt = - \int \frac{u^2}{u^2-1} du$$

$$\Rightarrow \frac{1}{2} t^{1/2} = - \int \frac{(u^2 - 1 + 1)}{u^2 - 1} du \quad (1)$$

$$\Rightarrow t^{1/2} = - \int \frac{u^2 - 1}{u^2 - 1} du - \int \frac{1}{u^2 - 1} du$$

$$\Rightarrow \sqrt{1 + y^2} = - \int du - \int \frac{1}{u^2 - (1)^2} du$$

[$\because 1 + y^2 = t$]

$$\Rightarrow \sqrt{1 + y^2} = -u - \frac{1}{2} \log \left| \frac{u - 1}{u + 1} \right| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + C \right]$$

$$\Rightarrow \sqrt{1 + y^2} = -\sqrt{1 + x^2} - \frac{1}{2} \log \left| \frac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2} + 1} \right| + C$$

which is the required solution. (1)

41. Find the particular solution of the differential equation satisfying the given condition

$$x^2 dy + (xy + y^2) dx = 0, \text{ when } y(1) = 1.$$

Delhi 2010

Given differential equation is

$$x^2 dy + (xy + y^2) dx = 0$$

Since, degree of each term is same, so the above equation is a homogeneous equation.

This equation can be written as

$$x^2 dy = -(xy + y^2) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(xy + y^2)}{x^2} \quad \dots(i)$$

On putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq (i), we get

$$v + x \frac{dv}{dx} = \frac{-(vx^2 + v^2x^2)}{x^2} = -(v + v^2)$$

$$\Rightarrow x \frac{dv}{dx} = -v - v^2 - v$$

$$\Rightarrow x \frac{dv}{dx} = -v^2 - 2v$$

$$\Rightarrow \frac{dv}{v^2 + 2v} = \frac{-dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{dv}{v^2 + 2v} = - \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^2 + 2v + 1 - 1} = - \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{(v+1)^2 - (1)^2} = - \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{v+1-1}{v+1+1} \right| = - \log |x| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{v}{v+2} \right| = - \log |x| + C$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{\frac{y}{x}}{\frac{y}{x} + 2} \right| = - \log |x| + C, \quad \left[\begin{array}{l} \because y = vx \\ \therefore v = \frac{y}{x} \end{array} \right]$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{y}{y+2x} \right| = - \log |x| + C \quad \dots(ii)$$

Also, given that $y = 0$ at $x = 1, y = 1$.

On putting $x = y = 1$ in Eq. (ii), we get

$$\therefore \frac{1}{2} \log \left| \frac{1}{1+2} \right| = - \log 1 + C$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{1}{3} \right| = -\log 1 + C$$

$$\Rightarrow C = \frac{1}{2} \log \frac{1}{3} \quad [:\because \log 1 = 0] \quad (1)$$

On putting the value of C in Eq. (ii), we get

$$\frac{1}{2} \log \left| \frac{y}{y+2x} \right| = -\log |x| + \frac{1}{2} \log \frac{1}{3}$$

$$\Rightarrow \log \left| \frac{y}{y+2x} \right| = -2 \log |x| + \log \frac{1}{3}$$

$$\Rightarrow \log \frac{y}{y+2x} = \log x^{-2} + \log \frac{1}{3}$$

$$[:\because n \log m = \log m^n]$$

$$\Rightarrow \log \frac{y}{y+2x} = \log \frac{1}{x^2} + \log \frac{1}{3}$$

$$\Rightarrow \log \left(\frac{y}{y+2x} \right) = \log \frac{1}{3x^2}$$

$$[:\because \log m + \log n = \log mn]$$

$$\Rightarrow \frac{y}{y+2x} = \frac{1}{3x^2}$$

$$\Rightarrow y \cdot 3x^2 = y + 2x$$

$$\Rightarrow y(1 - 3x^2) = -2x$$

$$\therefore y = \frac{2x}{3x^2 - 1}$$

which is the required particular solution. **(1)**

42. Find the particular solution of the differential equation satisfying the given condition

$$\frac{dy}{dx} = y \tan x, \text{ given that } y = 1, \text{ when } x = 0.$$

Given differential equation is

$$\frac{dy}{dx} = y \tan x$$

It can be written as $\frac{dy}{y} = \tan x \, dx$ (1)

On integrating both sides, we get

$$\int \frac{dy}{y} = \int \tan x \, dx$$

$$\Rightarrow \log|y| = \log|\sec x| + C \quad \dots(i) \quad (1)$$

$$\left[\because \int \frac{1}{y} \, dy = \log|y| \text{ and } \int \tan x \, dx = \log|\sec x| \right]$$

Also, given that $y = 1$, when $x = 0$.

On putting $x = 0$ and $y = 1$ in Eq.(i), we get

$$\log 1 = \log(\sec 0^\circ) + C$$

$$\Rightarrow 0 = \log 1 + C \quad [\because \sec 0^\circ = 1] \quad (1)$$

$$\Rightarrow C = 0 \quad [\because \log 1 = 0]$$

On putting $C = 0$ in Eq. (i), we get the required particular solution as

$$\log|y| = \log|\sec x|$$

$$\therefore y = \sec x \quad (1)$$

which is the required solution.

43. Solve the following differential equation

$$\cos^2 x \frac{dy}{dx} + y = \tan x.$$

All India 2009; Delhi 2008, 2011, 2008C

Given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

On dividing both sides by $\cos^2 x$, we get

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$$\Rightarrow \frac{dy}{dx} + y \cdot \sec^2 x = \tan x \cdot \sec^2 x \quad \dots(i)$$

$$\left[\because \frac{1}{\cos^2 x} = \sec^2 x \right]$$

which is the linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \sec^2 x \text{ and } Q = \tan x \cdot \sec^2 x \quad (1)$$

$$\therefore \text{IF} = e^{\int \sec^2 x \, dx} = e^{\tan x}$$

$$[\because \int \sec^2 x \, dx = \tan x + C] \quad (1)$$

Now, solution of linear differential equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C$$

$$\therefore y \times e^{\tan x} = \int \tan x \cdot \sec^2 x \cdot e^{\tan x} \, dx \dots(iii)$$

On putting $\tan x = t$

$$\Rightarrow \sec^2 x \, dx = dt \text{ in Eq. (iii), we get}$$

$$\therefore ye^{\tan x} = \int t e^t \, dt \quad (1)$$

$$\Rightarrow ye^{\tan x} = t \int e^t \, dt - \int \left[\frac{d}{dt}(t) \int e^t \, dt \right] dt$$

[using integration by parts in $\int t e^t \, dt$]

$$\Rightarrow ye^{\tan x} = t e^t - \int 1 \times e^t \, dt$$

$$\Rightarrow ye^{\tan x} = t e^t - e^t + C$$

$$\therefore ye^{\tan x} = \tan x \cdot e^{\tan x} - e^{\tan x} + C [\because \tan x = t]$$

On dividing both sides by $e^{\tan x}$, we get

$$y = \tan x - 1 + Ce^{-\tan x}$$

which is the required solution. (1)

44. Solve the following differential equation

$$\sec x \frac{dy}{dx} - y = \sin x. \quad \text{All India 2009C}$$

Given differential equation is

$$\sec x \frac{dy}{dx} - y = \sin x$$

On dividing both sides by $\sec x$, we get

$$\frac{dy}{dx} - \frac{y}{\sec x} = \frac{\sin x}{\sec x}$$

$$\Rightarrow \frac{dy}{dx} - y \cos x = \sin x \cos x \quad \dots(i)$$

which is a linear differential equation of the

$$\text{form} \quad \frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = -\cos x \text{ and } Q = \sin x \cos x \quad (1)$$

$$\therefore \text{IF} = e^{\int -\cos x \, dx} = e^{-\sin x}$$

$$[\because \int \cos x \, dx = \sin x + C] \quad (1)$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C$$

$$\therefore ye^{-\sin x} = \int \sin x \cos x e^{-\sin x} \, dx$$

On putting $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\therefore ye^{-\sin x} = \int t e^{-t} \, dt \quad (1)$$

$$\Rightarrow ye^{-\sin x} = t \int e^{-t} \, dt - \int \left[\frac{d}{dt}(t) \int e^{-t} \, dt \right] dt$$

[using integration by parts]

$$\Rightarrow ye^{-\sin x} = -te^{-t} - \int 1 \times (-e^{-t}) \, dt$$

$$= -te^{-t} + \int e^{-t} \, dt$$

$$\Rightarrow ye^{-\sin x} = -te^{-t} - e^{-t} + C$$

$$\Rightarrow ye^{-\sin x} = -\sin x e^{-\sin x} - e^{-\sin x} + C$$

[$\because \sin x = t$]

$$y = -\sin x - 1 + C e^{\sin x} \quad (1)$$

which is the required solution.

45. Solve the following differential equation

$$(x \log x) \frac{dy}{dx} + y = 2 \log x. \quad \text{Delhi 2009, 2009C}$$

Given differential equation is

$$(x \log x) \frac{dy}{dx} + y = 2 \log x$$

On dividing both sides by $x \log x$, we get

$$\frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x} \quad \dots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{x \log x} \text{ and } Q = \frac{2}{x} \quad (1)$$

$$\begin{aligned} \therefore \text{IF} &= e^{\int \frac{1}{x \log x} dx} = e^{\log \log x} \\ &= \log x \quad [\because e^{\log x} = x] \quad (1) \end{aligned}$$

$$\left[\because \int \frac{1}{x \log x} dx \Rightarrow \text{put } \log x = t \Rightarrow \frac{1}{x} dx = dt \right]$$

$$\therefore \int \frac{1}{x \log x} dx = \int \frac{dt}{t} = \log |t| = \log |\log x|$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore y \times \log x = \int \frac{2}{x} \log x dx \quad (1)$$

$$\begin{aligned} \Rightarrow y \log x &= \log x \int \frac{2}{x} dx \\ &\quad - \int \left[\frac{d}{dx} (\log x) \int \frac{2}{x} dx \right] dx \end{aligned}$$

[using integration by parts]

$$\Rightarrow y \log x = \log x \cdot 2 \log x - \int \frac{1}{x} \cdot 2 \log x \, dx$$

$$\left[\because \int \frac{1}{x} \, dx = \log |x| + C \right]$$

$$\Rightarrow y \log x = 2 (\log x)^2 - 2 \int \frac{\log x}{x} \, dx$$

$$\Rightarrow y \log x = 2 (\log x)^2 - \frac{2(\log x)^2}{2} + C$$

$$\left[\text{in } \int \frac{\log x}{x} \, dx, \text{ put } \log x = t \Rightarrow \frac{1}{x} \, dx = dt \right]$$

$$\therefore \int t \, dt = \frac{t^2}{2} = \frac{(\log x)^2}{2} + C$$

$$y = 2 (\log x) - (\log x) + \frac{C}{\log x}$$

[dividing both sides by $\log x$] (1)
which is the required solution.

46. Solve the following differential equation

$$x \frac{dy}{dx} = y - x \tan \left(\frac{y}{x} \right). \quad \text{All India 2009}$$

Given differential equation is

$$x \frac{dy}{dx} = y - x \tan \left(\frac{y}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x \tan \left(\frac{y}{x} \right)}{x} \quad \dots(i)$$

which is a homogeneous differential equation.

On putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{vx - x \tan v}{x} = v - \tan v$$

$$\Rightarrow x \frac{dv}{dx} = -\tan v$$

$$\Rightarrow \frac{dv}{\tan v} = -\frac{dx}{x} \quad (1)$$

$$\Rightarrow \cot v \, dv = -\frac{dx}{x} \quad \left[\because \frac{1}{\tan v} = \cot v \right] (1)$$

On integrating both sides, we get

$$\int \cot v \, dv = -\int \frac{dx}{x}$$

$$\Rightarrow \log |\sin v| = -\log |x| + C$$

$$[\because \int \cot v \, dv = \log |\sin v| + C]$$

$$\Rightarrow \log |\sin v| + \log |x| = C$$

$$\Rightarrow \log |x \sin v| = C$$

$$[\because \log m + \log n = \log mn]$$

$$\therefore \log \left| x \sin \frac{y}{x} \right| = C \quad \left[\because v = \frac{y}{x} \right] (1)$$

which is the required solution.

47. Solve the following differential equation

$$(1 + x^2) \frac{dy}{dx} + y = \tan^{-1} x.$$

Delhi 2009

The given differential equation is

$$(1+x^2) \frac{dy}{dx} + y = \tan^{-1} x$$

On dividing both sides by $(1+x^2)$, we get

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{\tan^{-1} x}{1+x^2} \quad \dots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = \frac{1}{1+x^2} \text{ and } Q = \frac{\tan^{-1} x}{1+x^2} \quad (1)$$

$$\therefore \text{IF} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

$$\left[\because \int \frac{1}{1+x^2} dx = \tan^{-1} x + C \right] (1)$$

Now, solution of above equation is given by

$$y \times IF = \int (Q \times IF) dx + C$$

$$\therefore y \times e^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1+x^2} \cdot e^{\tan^{-1} x} dx \quad \dots(iii)$$

On putting $\tan^{-1} x = t$

$$\Rightarrow \frac{1}{1+x^2} dx = dt \quad (1)$$

in Eq. (iii), we get

$$ye^{\tan^{-1} x} = \int t e^t dt$$

$$\Rightarrow ye^{\tan^{-1} x} = t \int e^t dt - \int \left[\frac{d}{dt} (t) \int e^t dt \right] dt$$

[using integration by parts]

$$\Rightarrow ye^{\tan^{-1} x} = te^t - \int 1 \times e^t dt$$

$$\Rightarrow ye^{\tan^{-1} x} = te^t - e^t + C$$

$$\Rightarrow ye^{\tan^{-1} x} = \tan^{-1} x \cdot e^{\tan^{-1} x} - e^{\tan^{-1} x} + C$$

On dividing both sides by $e^{\tan^{-1} x}$, we get

$$y = \tan^{-1} x - 1 + Ce^{-\tan^{-1} x} \quad (1)$$

which is the required solution.

48. Solve the following differential equation

$$\frac{dy}{dx} + y = \cos x - \sin x.$$

Delhi 2009

Given differential equation is

$$\frac{dy}{dx} + y = \cos x - \sin x \quad \dots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = 1 \text{ and } Q = \cos x - \sin x \quad (1)$$

$$\therefore \text{IF} = e^{\int 1 dx} = e^x \quad (1)$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore ye^x = \int e^x (\cos x - \sin x) dx$$

$$\Rightarrow ye^x = \int e^x \cos x dx - \int e^x \sin x dx$$

$$\Rightarrow ye^x = \left[\cos x \int e^x dx - \int \left\{ \frac{d}{dx} (\cos x) \int e^x dx \right\} dx - \int e^x \sin x dx \right]$$

[applying integration by parts in the first integral]

$$\Rightarrow ye^x = [e^x \cos x - \int -\sin x \cdot e^x dx] - \int e^x \sin x dx \quad (1)$$

$$\Rightarrow ye^x = e^x \cos x + \int e^x \sin x dx - \int e^x \sin x dx + C$$

$$\Rightarrow ye^x = e^x \cos x + C$$

On dividing both sides by e^x , we get

$$y = \cos x + Ce^{-x}$$

which is the required solution. (1)

49. Solve the following differential equation

$$\frac{dy}{dx} + 2y \tan x = \sin x.$$

All India 2008

Given differential equation is

$$\frac{dy}{dx} + 2y \tan x = \sin x \quad \dots(i)$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$P = 2 \tan x \text{ and } Q = \sin x \quad (1)$$

$$\therefore \text{IF} = e^{\int 2 \tan x \, dx} = e^{2 \log |\sec x|}$$

$$= e^{\log \sec^2 x} = \sec^2 x \quad (1)$$

Now, solution of above equation is given by

$$y \times \text{IF} = \int (Q \times \text{IF}) \, dx + C$$

$$y \sec^2 x = \int \sin x \cdot \sec^2 x \, dx$$

$$\Rightarrow y \sec^2 x = \int \frac{\sin x}{\cos^2 x} \, dx \quad (1)$$

$$\Rightarrow y \sec^2 x = \int \sec x \tan x \, dx$$

$$\left[\because \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x \right]$$

$$\Rightarrow y \sec^2 x = \sec x + C$$

$$[\because \int \sec x \tan x \, dx = \sec x + C]$$

$$\therefore y = \frac{1}{\sec x} + \frac{C}{\sec^2 x}$$

$$\Rightarrow y = \cos x + C \cos^2 x \quad (1)$$

which is the required solution.

50. Solve the following differential equation

$$x^2 \frac{dy}{dx} = y^2 + 2xy.$$

All India 2008

Given differential equation is

$$x^2 \frac{dy}{dx} = y^2 + 2xy$$

which is a homogeneous differential equation as degree of each term is same in the equation.

Above equation can be written as

$$\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2} \quad \dots(i)$$

On putting

$$y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{v^2 x^2 + 2vx^2}{x^2} = v^2 + 2v$$

$$\Rightarrow v + x \frac{dv}{dx} = v^2 + 2v$$

$$\Rightarrow x \frac{dv}{dx} = v^2 + 2v - v \Rightarrow x \frac{dv}{dx} = v^2 + v$$

$$\Rightarrow \frac{dv}{v^2 + v} = \frac{dx}{x} \quad (1)$$

On integrating both sides, we get

$$\int \frac{dv}{v^2 + v} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v^2 + v + \frac{1}{4} - \frac{1}{4}} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{\left(v + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2 \times \frac{1}{2}} \log \left| \frac{v + \frac{1}{2} - \frac{1}{2}}{v + \frac{1}{2} + \frac{1}{2}} \right| = \log |x| + C$$

$$\left[\dots \int \frac{dx}{x} = \frac{1}{1} \log |x - a| \right] \quad (1)$$

$$\begin{aligned} & \left[\int \frac{1}{x^2 - a^2} - \frac{2a}{x^2 - a^2} \log|x+a| \right] \\ \Rightarrow & \log \left| \frac{v}{v+1} \right| - \log|x| = C \\ \Rightarrow & \log \left| \frac{v}{(v+1) \cdot x} \right| = C \\ & \left[\because \log m - \log n = \log \left(\frac{m}{n} \right) \right] \\ \Rightarrow & \log \left| \frac{\frac{y}{x}}{\left(\frac{y}{x} + 1 \right) x} \right| = C \quad \left[\begin{array}{l} \because y = vx \\ \therefore v = \frac{y}{x} \end{array} \right] \\ \Rightarrow & \log \left| \frac{y}{xy + x^2} \right| = C \quad (1) \end{aligned}$$

which is the required solution.

- 51.** Solve the following differential equation
 $(x^2 - y^2) dx + 2xy dy = 0$, given that $y = 1$,
when $x = 1$. **Delhi 2008**

Given differential equation is

$$(x^2 - y^2) dx + 2xy dy = 0$$

which is a homogeneous differential equation as degree of each term is same.

Above equation can be written as

$$(x^2 - y^2) dx = -2xy dy \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \dots(i)$$

$$\text{On putting } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq. (i), we get

$$\begin{aligned} v + x \frac{dv}{dx} &= \frac{v^2 x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v} \\ \Rightarrow x \frac{dv}{dx} &= \frac{v^2 - 1}{2v} - v \end{aligned}$$

$$= \frac{v^2 - 1 - 2v^2}{2v} = \frac{-1 - v^2}{2v} \quad (1)$$

$$\Rightarrow \frac{2v}{v^2 + 1} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{2v}{v^2 + 1} dv = -\int \frac{dx}{x}$$

On putting $v^2 + 1 = t \Rightarrow 2v dv = dt$

$$\therefore \int \frac{dt}{t} = -\log|x| + C$$

$$\Rightarrow \log|t| = -\log|x| + C$$

$$\Rightarrow \log|v^2 + 1| + \log|x| = C \quad [\because t = v^2 + 1]$$

$$\Rightarrow \log\left|\frac{y^2}{x^2} + 1\right| + \log|x| = C \quad \dots(ii)$$

$$\left[\because v = \frac{y}{x}\right] (1)$$

Also, given that $y = 1$, when $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (ii), we get

$$\log 2 + \log 1 = C \Rightarrow C = \log 2 \quad [\because \log 1 = 0]$$

On putting $C = \log 2$ in Eq. (ii), we get

$$\log\left|\frac{y^2 + x^2}{x^2}\right| + \log x = \log 2$$

$$\Rightarrow \log\left|x\left(\frac{x^2 + y^2}{x^2}\right)\right| = \log 2$$

$$[\because \log m + \log n = \log mn]$$

$$\Rightarrow \log\left|\frac{x^2 + y^2}{x}\right| = \log 2 \Rightarrow x^2 + y^2 = 2x \quad (1)$$

which is the required solution.

52. Solve the following differential equation

$$\frac{dy}{dx} = \frac{x(2y - x)}{x(2y + x)}, \text{ if } y = 1, \text{ when } x = 1.$$

Delhi 2008

Given differential equation is

$$\frac{dy}{dx} = \frac{x(2y - x)}{x(2y + x)} \Rightarrow \frac{dy}{dx} = \frac{2xy - x^2}{2xy + x^2} \quad \dots(i)$$

which is a homogeneous differential equation because each term of numerator and denominator have same degree.

$$\text{On putting } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \quad (1)$$

in Eq. (i), we get

$$v + x \frac{dv}{dx} = \frac{2vx^2 - x^2}{2vx^2 + x^2} = \frac{2v - 1}{2v + 1}$$

$$v + x \frac{dv}{dx} = \frac{2v - 1}{2v + 1}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v - 1}{2v + 1} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{2v - 1 - 2v^2 - v}{2v + 1}$$

$$\frac{2v + 1}{2v^2 - v + 1} dv = - \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{2v + 1}{2v^2 - v + 1} dv = - \int \frac{dx}{x}$$

$$\Rightarrow I = - \log |x| + C \quad \dots(ii)$$

$$\text{where, } I = \int \frac{2v + 1}{2v^2 - v + 1} dv$$

$$\text{Let } 2v + 1 = A \cdot \frac{d}{dv} (2v^2 - v + 1) + B$$

$$\Rightarrow 2v + 1 = A(4v - 1) + B \quad \dots(iii)$$

On comparing coefficients of v and constants from both sides, we get

$$4A = 2$$

$$\Rightarrow A = \frac{1}{2} \text{ and } -A + B = 1$$

$$\Rightarrow -\frac{1}{2} + B = 1 \Rightarrow B = \frac{3}{2}$$

On putting $A = \frac{1}{2}$ and $B = \frac{3}{2}$ in Eq. (iii), we get

$$2v + 1 = \frac{1}{2}(4v - 1) + \frac{3}{2} \quad (1)$$

On integrating both sides, we get

$$I = \int \frac{2v + 1}{2v^2 - v + 1} dv$$

$$\Rightarrow I = \int \frac{\frac{1}{2}(4v - 1) + \frac{3}{2}}{2v^2 - v + 1} dv$$

$$\Rightarrow I = \frac{1}{2} \int \frac{4v - 1}{2v^2 - v + 1} dv + \frac{3}{2} \int \frac{dv}{2v^2 - v + 1}$$

$$\Rightarrow I = \frac{1}{2} \log |2v^2 - v + 1| + \frac{3}{4} \int \frac{dv}{v^2 - \frac{v}{2} + \frac{1}{2}}$$

$$\left[\begin{array}{l} \because \int \frac{4v - 1}{2v^2 - v + 1} dv \Rightarrow \text{put } 2v^2 - v + 1 = t \\ \qquad \qquad \qquad \Rightarrow (4v - 1) dv = dt \\ \text{then } \int \frac{dt}{t} = \log |t| = \log |2v^2 - v + 1| \end{array} \right]$$

$$\Rightarrow I = \frac{1}{2} \log |2v^2 - v + 1|$$

$$+ \frac{3}{4} \int \frac{dv}{v^2 - \frac{1}{2}v + \frac{1}{2} + \frac{1}{16} - \frac{1}{16}}$$

$$= \frac{1}{2} \log |2v^2 - v + 1| + \frac{3}{4} \int \frac{dv}{\left(v - \frac{1}{4}\right)^2 + \frac{7}{16}}$$

$$= \frac{1}{2} \log |2v^2 - v + 1| + \frac{3}{4} \int \frac{dv}{\left(v - \frac{1}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2} \quad (1/2)$$

$$\Rightarrow I = \frac{1}{2} \log |2v^2 - v + 1| + \frac{3}{4} \times \frac{4}{\sqrt{7}} \tan^{-1} \left(\frac{v - \frac{1}{4}}{\frac{\sqrt{7}}{4}} \right)$$

$$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C \right]$$

$$\Rightarrow I = \frac{1}{2} \log |2v^2 - v + 1| + \frac{3}{\sqrt{7}} \tan^{-1} \left(\frac{4v - 1}{\sqrt{7}} \right)$$

On putting the value of I in Eq. (ii), we get

$$\frac{1}{2} \log |2v^2 - v + 1| + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{4v - 1}{\sqrt{7}} \right) = -\log |x| + C \quad (1/2)$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{2y^2}{x^2} - \frac{y}{x} + 1 \right| + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{\frac{4y}{x} - 1}{\sqrt{7}} \right) = -\log |x| + C \quad \left[\because \text{put } v = \frac{y}{x} \right]$$

$$\Rightarrow \frac{1}{2} \log \left| \frac{2y^2}{x^2} - \frac{y}{x} + 1 \right| + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{4y - x}{\sqrt{7} \cdot x} \right) = -\log |x| + C \quad \dots(iv)$$

Also, given that $y = 1$, when $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (iv), we get

$$\frac{1}{2} \log |2| + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{3}{\sqrt{7}} \right) = -\log 1 + C$$

$$\rightarrow \frac{1}{2} \log 2 + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{3}{\sqrt{7}} \right) = C \quad [\because \log 1 = 0]$$

$$\rightarrow \frac{1}{2} \log 2 + \frac{1}{2} \log 2 + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{3}{\sqrt{7}} \right)$$

On putting the value of C in Eq. (iv), we get

$$\frac{1}{2} \log \left(\frac{2y^2 - xy + x^2}{x^2} \right) + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{4y - x}{\sqrt{7}x} \right)$$

$$= -\log |x| + \frac{1}{2} \log 2 + \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{3}{\sqrt{7}} \right)$$

$$\Rightarrow \log \left(\frac{2y^2 - xy + x^2}{x^2} \right)^{1/2} + \log x - \log (2)^{1/2}$$

$$= \frac{3\sqrt{7}}{7} \left[\tan^{-1} \left\{ \frac{\frac{3}{\sqrt{7}} - \left(\frac{4y - x}{\sqrt{7}x} \right)}{1 + \frac{3}{\sqrt{7}} \cdot \left(\frac{4y - x}{\sqrt{7}x} \right)} \right\} \right]$$

$$\left[\because \tan^{-1} A - \tan^{-1} B = \tan^{-1} \left(\frac{A - B}{1 + AB} \right) \right] \quad (1/2)$$

$$\Rightarrow \log (2y^2 - xy + x^2)^{1/2} - \log \sqrt{2}$$

$$= \frac{3\sqrt{7}}{7} \tan^{-1} \left[\frac{(4x - 4y) \cdot \sqrt{7}}{4x + 12y} \right]$$

$$\left[\because \log \left(\frac{2y^2 - xy + x^2}{x^2} \right) = \log (2y^2 - xy + x^2)^{1/2} \right. \\ \left. = \log (2y^2 - xy + x^2)^{1/2} - \log x \right]$$

$$\Rightarrow \log \sqrt{\frac{2y^2 - xy + x^2}{2}}$$

$$= \frac{3\sqrt{7}}{7} \tan^{-1} \left(\frac{(x - y) \cdot \sqrt{7}}{x + 3y} \right)$$

$$\begin{aligned} \Rightarrow \log \sqrt{\frac{2y^2 - xy + x^2}{2}} \\ = \frac{3\sqrt{7}}{7} \tan^{-1} \left[\frac{\sqrt{7}x - \sqrt{7}y}{x + 3y} \right] \end{aligned} \quad (1/2)$$

which is the required solution.

6 Marks Questions

- 53.** Find the particular solution of the differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$, for $x = 1$ and $y = 1$. Delhi 2013C

Given differential equation is

$$(3x^2 + y^2)dx + (x^2 + xy)dy = 0$$

It can be rewritten as $\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}$... (i)

which is a homogeneous differential equation of degree 2.

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

in Eq.(i), we get $v + x \frac{dv}{dx} = -\frac{3vx^2 + v^2x^2}{x^2 + vx^2}$

$$\Rightarrow x \frac{dv}{dx} = -\frac{3v + v^2}{1 + v} - v$$

$$\Rightarrow x \frac{dv}{dx} = -\left(\frac{3v + v^2 + v + v^2}{1 + v} \right) \quad (1)$$

$$\Rightarrow x \frac{dv}{dx} = -\left(\frac{2v^2 + 4v}{1 + v} \right) \Rightarrow \frac{(1 + v)dv}{2(v^2 + 2v)} = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{1 + v}{2(v^2 + 2v)} dv = -\int \frac{dx}{x} \quad \dots(ii) \quad (1)$$

Again, put $v^2 + 2v = z \Rightarrow (2v + 2)dv = dz$

$$\Rightarrow (1 + v)dv = \frac{dz}{2}$$

Then, Eq. (ii) becomes,

$$\int \frac{1}{2} \times \frac{dz}{2z} = -\int \frac{dx}{x} \quad (1)$$

$$\Rightarrow \frac{1}{4} \log|z| = -\log|x| + \log|C|$$

$$\Rightarrow \frac{1}{4} [\log|z| + 4\log|x|] = \log|C|$$

$$\Rightarrow \log|zx^4| = 4\log|C|$$

$$\Rightarrow zx^4 = C^4 = C_1 \quad zx^4 = C_1$$

where, $C_1 = C^4$

$$\Rightarrow x^4(v^2 + 2v) = C_1 \quad [\text{put } z = v^2 + 2v]$$

$$\Rightarrow x^4 \left(\frac{y^2}{x^2} + \frac{2y}{x} \right) = C_1 \left[\text{put } v = \frac{y}{x} \right] \dots (iii) \quad (1)$$

Also, given that $y = 1$ for $x = 1$.

On putting $x = 1$ and $y = 1$ in Eq. (iii), we get

$$1 \left(\frac{1}{1} + \frac{2}{1} \right) = C_1$$

$$\Rightarrow C_1 = 3 \quad (1)$$

Also, given that $y = 1$ for $x = 1$.

So, on putting $C_1 = 3$ in Eq. (iii), we get

$$x^4 \left(\frac{y^2}{x^2} + \frac{2y}{x} \right) = 3 \Rightarrow y^2x^2 + 2yx^3 = 3 \quad (1)$$

which is the required particular solution.

- 54.** Show that the differential equation $2ye^{x/y} dx + (y - 2xe^{x/y}) dy = 0$ is homogeneous. Find the particular solution of this differential equation, given that $x=0$, when $y=1$.

HOTS; Delhi 2013



Firstly, replace x by λx and y by λy in $f(x, y)$ of given differential equation to check that it is homogeneous. If it is homogeneous, then put $x = vy$ and $\frac{dx}{dy} = v + y \frac{dv}{dy}$ and then solve.

Given differential equation is

$2y e^{x/y} dx + (y - 2x e^{x/y}) dy = 0$. It can be written as

$$\frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}} \quad \dots(i)$$

$$\text{Let } F(x, y) = \frac{\left(2xe^{\frac{x}{y}} - y\right)}{\left(2ye^{\frac{x}{y}}\right)}$$

On replacing x by λx and y by λy both sides, we get

$$F(\lambda x, \lambda y) = \frac{\left(2\lambda x e^{\frac{\lambda x}{\lambda y}} - \lambda y\right)}{\left(2\lambda y e^{\frac{\lambda x}{\lambda y}}\right)}$$

$$\Rightarrow F(\lambda x, \lambda y) = \frac{\lambda(2xe^{x/y} - y)}{\lambda(2ye^{x/y})} = \lambda^0 [F(x, y)] \quad (1)$$

Thus, $F(x, y)$ is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation. (1)

To solve it, put $x = vy$

$$\Rightarrow \frac{dx}{dy} = v + y \frac{dv}{dy} \quad (1/2)$$

in Eq.(i), we get $v + y \frac{dv}{dy} = \frac{2ve^v - 1}{2e^v}$

$$\Rightarrow y \frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v = \frac{2ve^v - 1 - 2ve^v}{2e^v}$$

$$\Rightarrow 2e^v dv = \frac{-dy}{y} \quad (1)$$

On integrating both sides, we get

$$\int 2e^v dv = - \int \frac{dy}{y} \Rightarrow 2e^v = -\log|y| + C$$

Now, replace v by $\frac{x}{y}$, we get

$$2e^{x/y} + \log|y| = C \quad \dots(ii) \quad (1\frac{1}{2})$$

Also, given that $x = 0$, when $y = 1$.

On substituting $x = 0$ and $y = 1$ in Eq. (ii), we get

$$2e^0 + \log|1| = C \Rightarrow C = 2$$

On substituting the value of C in Eq. (ii), we get

$$2e^{x/y} + \log|y| = 2$$

which is the required particular solution of the given differential equation. (1)

55. Show that the differential equation

$$x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0 \text{ is homogeneous.}$$

Find the particular solution of this differential

equation, given that $x = 1$, when $y = \frac{\pi}{2}$. Delhi 2013

Given differential equation is

$$x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) = y \sin\left(\frac{y}{x}\right) - x \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}} \quad \dots(i)$$

$$\left[\text{dividing both sides by } x \sin\left(\frac{y}{x}\right) \right]$$

Let $(x, y) = \frac{y}{x} - \frac{1}{\sin\frac{y}{x}}$

On replacing x by λx and y by λy on both sides, we get

$$\begin{aligned} F(\lambda x, \lambda y) &= \frac{\lambda y}{\lambda x} - \frac{1}{\sin\frac{\lambda y}{\lambda x}} = \lambda^0 \left(\frac{y}{x} - \frac{1}{\sin\frac{y}{x}} \right) \\ &= \lambda^0 F(x, y) \end{aligned}$$

So, given differential equation is homogeneous.

(2)

On putting $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} \text{ in Eq.(i), we get} \quad (1)$$

$$v + x \frac{dv}{dx} = v - \frac{1}{\sin v}$$

$$\Rightarrow x \frac{dv}{dx} = -\frac{1}{\sin v} \Rightarrow \sin v \, dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\int \sin v \, dv = -\int \frac{dx}{x}$$

$$\Rightarrow -\cos v = -\log|x| + C$$

$$\Rightarrow -\cos y/x = -\log|x| + C \left[\because v = \frac{y}{x} \right] \quad (1\frac{1}{2}) \dots (ii)$$

Also, given that $x = 1$, when $y = \frac{\pi}{2}$.

On putting $x = 1$ and $y = \frac{\pi}{2}$ in Eq. (ii), we get

$$-\cos\left(\frac{\pi}{2}\right) = -\log|1| + C$$

$$\Rightarrow -0 = -0 + C \Rightarrow C = 0$$

On putting the value of C in Eq. (ii), we get

$$\cos \frac{y}{x} = \ln|x|$$

which is the required solution. (1½)

56. Find the particular solution of the

differential equation $\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y$,

($y \neq 0$), given that $x=0$, when $y = \frac{\pi}{2}$.

All India 2013

Given differential equation is

$$\frac{dx}{dy} + x \cot y = 2y + y^2 \cot y, (y \neq 0)$$

which is a linear differential equation.

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \cot y \text{ and } Q = 2y + y^2 \cot y$$

$$\therefore \text{IF} = e^{\int P dy} = e^{\int \cot y dy} = e^{\log \sin y} = \sin y \quad \text{(1½)}$$

Now, the solution of above differential equation is given by

$$x \cdot (\text{IF}) = \int Q \cdot (\text{IF}) dy + C$$

$$\begin{aligned} \therefore x \sin y &= \int (2y + y^2 \cot y) \sin y dy + C \\ &= 2 \int y \sin y dy + \int y^2 \cos y dy + C \\ &= 2 \int y \sin y dy + y^2 \int \cos y dy \\ &\quad - \int \left[\left(\frac{d}{dy} y^2 \right) \int \cos y dy \right] dy + C \end{aligned}$$

[using integration by parts in second integral]

$$\begin{aligned} &= 2 \int y \sin y dy + y^2 \sin y - 2 \int y \sin y dy + C \\ &= y^2 \sin y + C \end{aligned}$$

$$\Rightarrow x \sin y = y^2 \sin y + C \quad \dots \text{(i) (2)}$$

Also, given that $x = 0$, when $y = \frac{\pi}{2}$.

On putting $x = 0$ and $y = \frac{\pi}{2}$ in Eq. (i), we get

$$0 = \left(\frac{\pi}{2}\right)^2 \sin \frac{\pi}{2} + C \Rightarrow C = -\frac{\pi^2}{4} \quad (1/2)$$

On putting the value of C in Eq. (i), we get

$$x \sin y = y^2 \sin y - \frac{\pi^2}{4} \Rightarrow x = y^2 - \frac{\pi^2}{4} \cdot \operatorname{cosec} y$$

which is required particular solution of given differential equation. (2)

57. Show that the differential equation

$$\left[x \sin^2 \left(\frac{y}{x} \right) - y \right] dx + x dy = 0 \text{ is homogeneous.}$$

Find the particular solution of this differential equation, given that

$$y = \frac{\pi}{4}, \text{ when } x = 1.$$

All India 2013

Given differential equation is

$$\left[x \sin^2\left(\frac{y}{x}\right) - y \right] dx + x dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x} \quad \dots(i)$$

Let $F(x, y) = \frac{y - x \sin^2\left(\frac{y}{x}\right)}{x}$

On replacing x by λx and y by λy both sides, we get

$$F(\lambda x, \lambda y) = \frac{\lambda \left[y - x \sin^2\left(\frac{y}{x}\right) \right]}{\lambda x} = \lambda^0 [F(x, y)]$$

Thus, given differential equation is a homogeneous differential equation. **(1)**

On putting $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$ in Eq. (i), we get

$$\begin{aligned} v + x \frac{dv}{dx} &= \frac{vx - x \sin^2\left(\frac{vx}{x}\right)}{x} \\ \Rightarrow v + x \frac{dv}{dx} &= v - \sin^2 v \Rightarrow x \frac{dv}{dx} = -\sin^2 v \\ \Rightarrow \operatorname{cosec}^2 v dv &= \frac{-dx}{x} \quad \dots(2) \end{aligned}$$

On intergrating both sides, we get

$$\begin{aligned} \int \operatorname{cosec}^2 v dv + \int \frac{dx}{x} &= 0 \\ \Rightarrow -\cot v + \log|x| &= C \\ \Rightarrow -\cot\left(\frac{y}{x}\right) + \log|x| &= C \left[\because v = \frac{y}{x} \right] \dots(ii) \end{aligned}$$

Also, given that, $y = \frac{\pi}{4}$, when $x = 1$.

On putting $x = 1$ and $y = \frac{\pi}{4}$, in Eq. (ii), we get

$$-\cot\left(\frac{\pi}{4}\right) + \log|1| = C \quad (2)$$

$$\Rightarrow C = -1 \quad \left[\because \cot\frac{\pi}{4} = 1 \right]$$

On putting the value of C in Eq. (ii), we get

$$-\cot\left(\frac{y}{x}\right) + \log|x| = -1$$

$$\Rightarrow 1 + \log|x| - \cot\left(\frac{y}{x}\right) = 0$$

which is the required particular solution of given differential equation. **(1)**

- 58.** Find the particular solution of the differential equation $(\tan^{-1} y - x)dy = (1 + y^2)dx$, given that $x=0$, when $y = 0$. All India 2013

Given differential equation is

$$(\tan^{-1} y - x)dy = (1 + y^2)dx$$

$$\Rightarrow \frac{\tan^{-1} y - x}{1 + y^2} = \frac{dx}{dy} \Rightarrow \frac{dx}{dy} = \frac{-x}{1 + y^2} + \frac{\tan^{-1} y}{1 + y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2} \cdot x = \frac{\tan^{-1} y}{1 + y^2}$$

which is a linear differential equation of first order. **(1)**

On comparing with $\frac{dx}{dy} + Px = Q$, we get

$$P = \frac{1}{1 + y^2} \quad \text{and} \quad Q = \frac{\tan^{-1} y}{1 + y^2}$$

$$\therefore \text{IF} = e^{\int P dy} = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y} \quad (1)$$

Now, solution of above differential equation is given by

$$\begin{aligned} x \cdot (\text{IF}) &= \int Q \cdot (\text{IF}) dy + C \\ \Rightarrow x e^{\tan^{-1} y} &= \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y} + C \quad (1) \end{aligned}$$

$$\text{On putting } t = \tan^{-1} y \Rightarrow dt = \frac{1}{1+y^2} dy$$

$$\begin{aligned} \therefore x \cdot e^{\tan^{-1} y} &= \int t \cdot e^t dt + C \\ \Rightarrow x \cdot e^{\tan^{-1} y} &= t \cdot e^t - \int 1 \cdot e^t dt + C \\ &\quad \text{[using integration by parts]} \\ \Rightarrow x \cdot e^{\tan^{-1} y} &= t \cdot e^t - e^t + C \\ \Rightarrow x \cdot e^{\tan^{-1} y} &= (\tan^{-1} y - 1) e^{\tan^{-1} y} + C \dots(i) \quad (1) \end{aligned}$$

Also, given that, when $x=0$, then $y=0$.

On putting $x=0, y=0$ in Eq. (i), we get

$$\begin{aligned} 0 &= (\tan^{-1} 0 - 1) e^{\tan^{-1} 0} + C \\ \Rightarrow 0 &= (0 - 1) e^0 + C \Rightarrow 0 = (0 - 1) \cdot 1 + C \\ \Rightarrow C &= 1 \quad (1) \end{aligned}$$

On putting the value of C in Eq. (i), we get

$$\begin{aligned} \therefore x \cdot e^{\tan^{-1} y} &= (\tan^{-1} y - 1) \cdot e^{\tan^{-1} y} + 1 \\ \Rightarrow x &= \tan^{-1} y - 1 + e^{-\tan^{-1} y} \end{aligned}$$

which is the required particular solution of the differential equation. (1)