Types of Integrals

1 Mark Questions

1. Find $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx.$ **Delhi 2014C**

Let $I = \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx - \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx$

$= \int \sec^2 dx - \int \operatorname{cosec}^2 x \, dx$

$= \tan x + \cot x + C$ **(1)**

2. Find $\int \frac{\sin^6 x}{\cos^8 x} dx.$ **All India 2014C**

Let $I = \int \frac{\sin^6 x}{\cos^8 x} dx = \int \tan^6 x \sec^2 x \, dx$

Put $\tan x = t$

$\Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int t^6 \, dt = \frac{t^7}{7} + C = \frac{\tan^7 x}{7} + C$ **(1)**

3. Evaluate $\int \frac{dx}{\sin^2 x \cos^2 x}$.

Delhi 2014C; Foreign 2014

> Firstly, divide numerator and denominator by $\cos^4 x$ and use $\sec^2 x = 1 + \tan^2 x$, then put $\tan x = t$ and integrate.

Let $I = \int \frac{dx}{\sin^2 \cos^2 x}$

On dividing the numerator and denominator by $\cos^4 x$, we get

$$I = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^2 x} dx$$

$$\Rightarrow \quad I = \int \frac{(1 + \tan^2 x) \cdot \sec^2 x}{\tan^2 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\therefore \quad I = \int \frac{1 + t^2}{t^2} dt = \int 1 dt + \int \frac{1}{t^2} dt$$

$$\Rightarrow \quad I = t - \frac{1}{t} + C$$

$$\Rightarrow \quad I = \tan x - \cot x + C$$ **(1)**

4. Evaluate $\int \cos^{-1}(\sin x) dx$. **Delhi 2014C**

Let $I = \int \cos^{-1}(\sin x)\, dx$

$$= \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx$$

$$= \int \left(\frac{\pi}{2} - x\right) dx \quad [\because \cos^{-1}(\cos\theta) = \theta]$$

$$= \frac{\pi}{2}\int dx - \int x\, dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$$

5. Write the anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$.

Delhi 2014

Anti-derivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$

$$= \int \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = 3\int \sqrt{x}\, dx + \int \frac{1}{\sqrt{x}} dx$$

$$= 3\left(\frac{x^{1/2+1}}{1/2+1}\right) + \left[\frac{x^{-1/2+1}}{-1/2+1}\right] + C$$

$$= 2(x^{3/2} + x^{1/2}) + C$$ **(1)**

6. Evaluate $\int (1-x)\sqrt{x}\, dx$.

HOTS; Delhi 2012

> Firstly, multiply the two functions and then use $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$.

Let $I = \int (1-x)\sqrt{x}\, dx = \int (\sqrt{x} - x\sqrt{x})\, dx$

$$= \int (x^{1/2} - x^{3/2})\, dx = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$$

$$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1\right]$$ **(1)**

7. Given, $\int e^x (\tan x + 1)\sec x dx = e^x f(x) + C$. Write $f(x)$ satisfying above.

All India 2012; Foreign 2011

> Use the relation $\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$ and simplify it.

Given that, $\int e^x (\tan x + 1) \sec x dx = e^x \cdot f(x) + C$

$$\Rightarrow \int e^x (\sec x + \sec x \tan x) dx = e^x f(x) + C$$

$$\Rightarrow \quad e^x \cdot \sec x + C = e^x f(x) + C$$

$$\left[\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x) \text{ and } \frac{d}{dx}(\sec x) = \sec x \tan x\right]$$

On comparing both sides, we get

$$f(x) = \sec x \qquad \textbf{(1)}$$

8. Evaluate $\int \frac{2}{1+\cos 2x} dx$.

Foreign 2012

Let $I = \int \frac{2}{1+\cos 2x} dx$

$$= \int \frac{2}{2\cos^2 x} dx \qquad [\because \cos 2\theta = 2\cos^2\theta - 1]$$

$$= \int \sec^2 x dx = \tan x + C \qquad \textbf{(1)}$$

9. Write the value of $\int \frac{x+\cos 6x}{3x^2+\sin 6x} dx$. **All India 2012C**

Let $I = \int \frac{x + \cos 6x}{3x^2 + \sin 6x} dx$

Put $3x^2 + \sin 6x = t$

$\Rightarrow \quad 6x + 6\cos 6x = \frac{dt}{dx} \Rightarrow (x + \cos 6x)\, dx = \frac{dt}{6}$

$\therefore I = \int \frac{dt}{6t} = \frac{1}{6} \log|t| + C \quad \left[\because \int \frac{1}{x} dx = \log|x|\right]$

$= \frac{1}{6} [\log|(3x^2 + \sin 6x)|] + C$ **(1)**

10. Write the value of $\int \frac{dx}{x^2 + 16}$.

Delhi 2011

Let $I = \int \frac{dx}{x^2 + 16} = \int \frac{dx}{x^2 + (4)^2}$

$= \frac{1}{4} \tan^{-1} \frac{x}{4} + C$

$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$ **(1)**

11. Write the value of $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$.

Delhi 2012C, 2011

Let $I = \int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx = \int \frac{\left(\frac{1}{\cos^2 x}\right)}{\left(\frac{1}{\sin^2 x}\right)} dx$

$= \int \frac{\sin^2 x}{\cos^2 x} dx$

$= \int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$[\because \tan^2 x + 1 = \sec^2 x]$

$= \int \sec^2 x\, dx - \int 1 dx = \tan x - x + C$

$[\because \int \sec^2 x\, dx = \tan x]$ **(1)**

12. Write the value of $\int \frac{2 - 3\sin x}{\cos^2 x}\,dx$.

Delhi 2011

Let $I = \int \frac{2 - 3\sin x}{\cos^2 x}\,dx = \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$

$= \int (2\sec^2 x - 3\sec x \tan x)dx$

$= 2\int \sec^2 x\,dx - 3\int \sec x \tan x\,dx$

$= 2\tan x - 3\sec x + C$ **(1)**

$\left[\because \int \sec^2 x\,dx = \tan x \text{ and } \int \sec x \tan x\,dx = \sec x\right]$

13. Evaluate $\int \frac{dx}{\sqrt{1 - x^2}}$.

All India 2011

Let $I = \int \frac{dx}{\sqrt{1 - x^2}} = \int \frac{dx}{\sqrt{(1)^2 - x^2}} = \sin^{-1} x + C$

$\left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}\right]$ **(1)**

14. Evaluate $\int \frac{(\log x)^2}{x}\,dx$.

All India 2011

> The differentiation of log x is in denominator. So, firstly put $\log x = t$ and adjust the integral in terms of t and then integrate.

Let $I = \int \frac{(\log x)^2}{x}\,dx$

Put $\log x = t \Rightarrow \frac{1}{x}\,dx = dt$

$\therefore I = \int \frac{(\log x)^2}{x}\,dx = \int t^2\,dt$

$= \frac{t^3}{3} + C = \frac{(\log x)^3}{3} + C$ **(1)**

15. Evaluate $\int \frac{e^{\tan^{-1} x}}{1+x^2} dx.$ **All India 2011**

Let $I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$\therefore$ $I = \int \frac{e^{\tan^{-1} x}}{1+x^2} dx = \int e^t \, dt$

$= e^t + C$ $\quad [\because \int e^x \, dx = e^x]$

$= e^{\tan^{-1} x} + C$ **(1)**

16. Evaluate $\int (ax+b)^3 dx.$ **All India 2011**

Let $I = \int (ax+b)^3 \, dx$

Put $t = ax+b$

$\Rightarrow$ $\frac{dt}{dx} = a \Rightarrow \frac{dt}{a} = dx$

$\therefore$ $I = \int \frac{t^3}{a} dt = \frac{1}{a} \cdot \frac{t^4}{4} + C$

$= \frac{(ax+b)^4}{4a} + C$ **(1)**

17. Evaluate $\int \frac{(1+\log x)^2}{x} dx.$

Foreign 2011; Delhi 2009

Let $I = \int \frac{(1+\log x)^2}{x} dx$

Put $1 + \log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore$ $I = \int \frac{(1+\log x)^2}{x} dx = \int t^2 \, dt = \frac{t^3}{3} + C$

$= \frac{(1+\log x)^3}{3} + C$ **(1)**

18. Evaluate $\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx.$

Foreign 2011

Let $I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx$

Put $e^{2x} + e^{-2x} = t$

$\Rightarrow \quad (2e^{2x} - 2e^{-2x})\, dx = dt \quad \left[\because \frac{d}{dx}(e^{ax}) = ae^{ax}\right]$

$\Rightarrow \quad (e^{2x} - e^{-2x})\, dx = \frac{dt}{2}$

$\therefore\ I = \int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \frac{1}{2} \int \frac{dt}{t}$

$= \frac{1}{2} \log |t| + C$

$= \frac{1}{2} \log | e^{2x} + e^{-2x} | + C$ **(1)**

19. Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx.$

Foreign 2011

Let $I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Put $\sqrt{x} = t$

$\Rightarrow \quad \frac{1}{2\sqrt{x}} dx = dt \quad \Rightarrow \quad \frac{1}{\sqrt{x}} dx = 2dt$

$\therefore \quad I = \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = 2 \int \cos t\, dt$

$= 2 \sin t + C = 2 \sin \sqrt{x} + C$ **(1)**

20. Evaluate $\int \frac{2 \cos x}{\sin^2 x} dx.$

All India 2011C, 2009, 2008

Let $I = \int \frac{2\cos x}{\sin^2 x} dx = \int 2 \operatorname{cosec} x \cot x \, dx$

$= -2\operatorname{cosec} x + C$ **(1)**

$[\because \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x]$

21. Evaluate $\int \frac{x^3 - x^2 + x - 1}{x - 1} dx.$ **Delhi 2011C**

Firstly, factorise numerator and cancel out common from numerator and denominator and then integrate.

Let $I = \int \frac{x^3 - x^2 + x - 1}{x - 1} dx$

$= \int \frac{x^2(x-1) + 1(x-1)}{x-1} dx$

$= \int \frac{(x^2+1)(x-1)}{x-1} dx = \int (x^2+1) \, dx$

$= \frac{x^3}{3} + x + C$ **(1)**

22. Write the value of $\int \frac{1 - \sin x}{\cos^2 x} dx.$ **All India 2011C**

Let $I = \int \frac{1 - \sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$

$= \int \sec^2 x \, dx - \int \sec x \tan x \, dx$

$= \tan x - \sec x + C$ **(1)**

23. Evaluate $\int \frac{2\cos x}{3\sin^2 x} dx.$ **All India 2011C**

Let $I = \int \frac{2\cos x}{3\sin^2 x} dx$

Let $\sin x = t \Rightarrow \cos x \, dx = dt$

$$\therefore \quad I = \int \frac{2dt}{3t^2} = \frac{2}{3}\int \frac{dt}{t^2} = \frac{2}{3} \frac{t^{-1}}{(-1)}$$

$$= \frac{-2}{3}(\sin x)^{-1} + C = \frac{-2}{3\sin x} + C \qquad \textbf{(1)}$$

Alternate Method

Let $I = \int \frac{2\cos x}{3\sin^2 x} dx = \frac{2}{3}\int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} dx$

$$= \frac{2}{3}\int \operatorname{cosec} x \cot x \, dx$$

$$= \frac{2}{3}(-\operatorname{cosec} x) + C = -\frac{2}{3\sin x} + C \qquad \textbf{(1)}$$

24. Evaluate $\int \frac{x^3 - 1}{x^2} dx.$

Delhi 2010C

Let $I = \int \frac{x^3 - 1}{x^2} dx = \int \left(\frac{x^3}{x^2} - \frac{1}{x^2}\right) dx$

$$= \int x \, dx - \int \frac{1}{x^2} dx = \frac{x^2}{2} - \frac{x^{-1}}{-1} + C$$

$$= \frac{x^2}{2} + \frac{1}{x} + C \qquad \textbf{(1)}$$

25. Evaluate $\int \sec^2 (7 - 4x) \, dx.$

Delhi 2010; All India 2010

Let $I = \int \sec^2 (7 - 4x) \, dx$

$$= \frac{\tan(7 - 4x)}{-4} + C$$

$$\left[\because \int \sec^2 ax \, dx = \frac{\tan ax}{a}\right]$$

$$= -\frac{\tan(7 - 4x)}{4} + C \qquad \textbf{(1)}$$

26. Evaluate $\int \frac{\log x}{x} dx$. All India 2010C

Let $I = \int \frac{\log x}{x} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$\therefore I = \int \frac{\log x}{x} dx = \int t\, dt = \frac{t^2}{2} + C$

$= \frac{(\log x)^2}{2} + C$ **(1)**

27. Evaluate $\int 2^x dx$. All India 2010C

Let $I = \int 2^x dx$

$= \frac{2^x}{\log 2} + C$ $\left[\because \int a^x dx = \frac{a^x}{\log a}\right]$

Alternate Method

Let $I = \int 2^x dx$

Again, let $2^x = t$

On taking log both sides, we get

$x \log 2 = \log t$

$\Rightarrow \log 2 dx = \frac{1}{t} dt \Rightarrow dx = \frac{1}{t \log 2} dt$

$\therefore I = \int \frac{t\, dt}{t \log 2} = \frac{1}{\log 2} \int dt$

$= \frac{t}{\log 2} + C = \frac{2^x}{\log 2} + C$ **(1)**

28. Evaluate $\int \frac{\log (\sin x)}{\tan x} dx$. HOTS; All India 2009C

The differentiation of log(sin x) is tan x, which exists as denominator. So, solve it by substitution method.

Let $$I = \int \frac{\log(\sin x)}{\tan x}\, dx$$

Put $\log(\sin x) = t$

$$\Rightarrow \quad \frac{1}{\sin x} \cdot \cos x\, dx = dt \Rightarrow \cot x\, dx = dt$$

$$\Rightarrow \quad \frac{1}{\tan x}\, dx = dt \quad \left[\because \cot x = \frac{1}{\tan x}\right]$$

$$\therefore \quad I = \int \frac{\log(\sin x)}{\tan x}\, dx = \int t\, dt$$

$$= \frac{t^2}{2} + C = \frac{(\log \sin x)^2}{2} + C \qquad \textbf{(1)}$$

29. Evaluate $\int (\operatorname{cosec}^2 x - \cot x)\, e^x\, dx$. **Delhi 2009C**

Let $I = \int (\operatorname{cosec}^2 x - \cot x)\, e^x\, dx$

$$= -\int (\cot x - \operatorname{cosec}^2 x)\, e^x\, dx$$

$$= -\int [\cot x + (-\operatorname{cosec}^2 x)]\, e^x\, dx$$

$$= -e^x \cot x + C$$

$$\left[\because \int e^x \{f(x) + f'(x)\}\, dx = e^x \cdot f(x)\right] \qquad \textbf{(1)}$$

30. Evaluate $\int \frac{\sec^2 \sqrt{x}}{\sqrt{x}}\, dx$. **All India 2009**

Do same as Que 19. [**Ans.** $2 \tan \sqrt{x} + C$]

31. Evaluate $\int \frac{\sin \sqrt{x}}{\sqrt{x}}\, dx$. **All India 2009**

Do same as Que 19. [**Ans.** $-2 \cos \sqrt{x} + C$]

32. Evaluate $\int \frac{\sec^2 x}{3 + \tan^2 x}\, dx$. **Delhi 2009**

Let $$I = \int \frac{\sec^2 x}{3 + \tan^2 x} dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{\sec^2 x}{3 + \tan^2 x} dx = \int \frac{dt}{3 + t^2} = \int \frac{dt}{(\sqrt{3})^2 + t^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C \left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x}{\sqrt{3}}\right) + C$$ **(1)**

33. Evaluate $\int \frac{x^2}{1 + x^3} dx.$ **Delhi 2009**

Let $$I = \int \frac{x^2}{1 + x^3} dx$$

Put $1 + x^3 = t \Rightarrow 3x^2\, dx = dt$

$$\Rightarrow \quad x^2\, dx = \frac{dt}{3}$$

$$\therefore \quad I = \int \frac{x^2}{1 + x^3} dx = \frac{1}{3} \int \frac{dt}{t}$$

$$= \frac{1}{3} \log|t| + C \qquad \left[\because \int \frac{dx}{x} = \log|x|\right]$$

$$= \frac{1}{3} \log|1 + x^3| + C$$ **(1)**

34. Evaluate $\int \sin^3 x dx.$ **HOTS; All India 2009**

Let $$I = \int \sin^3 x\, dx = \int \frac{3 \sin x - \sin 3x}{4} dx$$

$$= \frac{1}{4} \int 3 \sin dx - \frac{1}{4} \int \sin 3x\, dx$$

$$= \frac{1}{4}\left(-3 \cos x + \frac{\cos 3x}{3}\right) + C$$ **(1)**

35. Evaluate $\int \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\, dx.$ **All India 2009**

Let $I = \int \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\, dx = \int \sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\, dx$

$$\left[\because \cos 2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1\right]$$

$= \int \tan x\, dx = \log(\sec x) + C$ **(1)**

36. Evaluate $\int (x-3)\sqrt{x^2+3x-18}\, dx.$ **Delhi 2014**

> Here, integral is of the form $(px-q)\sqrt{ax^2+bx+c}$, so firstly write $x-3$ as $x-3 = A\frac{d}{dx}(x^2+3x-18)+B$ and find A and B.
> Then, integrate by using suitable method.

Let $I = \int (x-3)\sqrt{x^2+3x-18}\, dx$

Here, we can write $(x-3)$ as

$$x-3 = A\frac{d}{dx}(x^2+3x-18)+B$$

$\Rightarrow \quad x-3 = A(2x+3)+B$

On equating the coefficients of x and constant term from both sides, we get

$$2A = 1 \text{ and } 3A + B = -3$$

$\Rightarrow \quad A = \frac{1}{2}$ and $3 \times \frac{1}{2} + B = -3$

$\Rightarrow \quad A = \frac{1}{2}$ and $B = -\frac{3}{2} - 3$

$\Rightarrow \quad A = \frac{1}{2}$ and $B = -\frac{9}{2}$ **(1)**

Thus, the given integral reduces in the following form

$$I = \int \left\{\frac{1}{2}(2x+3) - \frac{9}{2}\right\} \sqrt{x^2+3x-18}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\, dx - \frac{9}{2}\int \sqrt{x^2+3x-18}\, dx$$

$$= \frac{1}{2} I_1 - \frac{9}{2} I_2 \qquad \ldots(i)$$

where, $I_1 = \int (2x+3)\sqrt{x^2+3x-18}\, dx$

Put $x^2 + 3x - 18 = t$

$\Rightarrow \quad (2x+3)\, dx = dt$

$$\therefore \quad I_1 = \int t^{1/2}\, dt = \frac{2}{3} t^{3/2} + C_1 \qquad \textbf{(1)}$$

$$= \frac{2}{3}(x^2+3x-18)^{3/2} + C_1$$

and $\quad I_2 = \int \sqrt{x^2+3x-18}\, dx$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - 18 - \frac{9}{4}}\, dx$$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \frac{81}{4}}\, dx$$

$$= \int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$$

$$= \frac{\left(x+\frac{3}{2}\right)}{2}\sqrt{x^2+3x-18}$$

$$-\frac{81}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|+C_2$$

$$\left[\because \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}|\right]$$

$$= \frac{2x+3}{4}\sqrt{x^2+3x-18}$$

$$-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|+C_2$$

(1)

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I = \frac{1}{2}\left[\frac{2}{3}(x^2+3x-18)^{3/2}+C_1\right]$$

$$-\frac{9}{2}\left[\frac{2x+3}{4}\sqrt{x^2+3x-18}\right.$$

$$\left.-\frac{81}{8}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|+C_2\right]$$

$$\Rightarrow \quad I = \frac{1}{3}(x^3+3x-18)^{3/2}$$

$$-\frac{9}{8}(2x+3)\sqrt{x^2+3x-18}$$

$$+\frac{729}{16}\log\left|\frac{2x+3}{2}+\sqrt{x^2+3x-18}\right|+C$$

(1)

where, $C = \frac{C_1}{2} - \frac{9C_2}{2}$

37. Evaluate $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx.$ **All India 2014**

Let $I = \int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$

Here, $(x+2)$ can be written as

$$x + 2 = A\frac{d}{dx}(x^2+5x+6) + B$$

$\Rightarrow$ $$x + 2 = A(2x+5) + B$$

On equating the coefficients of x and constant term from both sides, we get

$$2A = 1 \text{ and } 5A + B = 2$$

$\Rightarrow$ $A = \frac{1}{2}$ and then $B = -\frac{1}{2}$ **(1)**

$$\therefore \quad I = \int \frac{\left\{\frac{1}{2}(2x+5) - \frac{1}{2}\right\}}{\sqrt{x^2+5x+6}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}\,dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+5x+6}}\,dx$$

$\Rightarrow$ $I = \frac{1}{2}I_1 - \frac{1}{2}I_2$ ...(i)

where, $I_1 = \int \frac{2x+5}{\sqrt{x^2+5x+6}}\,dx$

Put $x^2 + 5x + 6 = t$

$\Rightarrow$ $(2x+5)\,dx = dt$

$$\therefore \quad I_1 = \int \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{x^2+5x+6} + C_1 \quad \text{...(ii)}$$ **(1)**

and $I_2 = \int \frac{1}{\sqrt{x^2+5x+6}}\,dx$

$$= \int \frac{1}{\sqrt{x^2 + 2 \times \frac{5}{2} \times x + 6 - \frac{25}{4} + \frac{25}{4}}} dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 + 6 - \frac{25}{4}}} dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} dx$$

$$= \log\left| \left(x + \frac{5}{2}\right) + \sqrt{\left(x + \frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + C_2$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right|\right]$$

$$\Rightarrow I_2 = \log\left| x + \frac{5}{2} + \sqrt{x^2 + 5x + 6} \right| + C_2 \quad \text{...(iii)}$$

(1)

On putting the values of I_1 and I_2 from Eqs. (ii) and (iii) in Eq. (i), we get

$$I = \frac{1}{2}[2\sqrt{x^2+5x+6} + C_1]$$

$$-\frac{1}{2}\left[\log\left| x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + C_2\right]$$

$$= \sqrt{x^2+5x+6} + \frac{C_1}{2}$$

$$-\frac{1}{2}\log\left| x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| - \frac{C_2}{2}$$

$$\Rightarrow \quad I = \sqrt{x^2+5x+6}$$

$$-\frac{1}{2}\log\left| x+\frac{5}{2}+\sqrt{x^2+5x+6}\right| + C \quad \textbf{(1)}$$

where, $C = \frac{C_1}{2} - \frac{C_2}{2}$

38. Evaluate $\int (3x-2)\sqrt{x^2+x+1}\,dx$.

Foreign 2014

Let $I = \int (3x - 2)\sqrt{x^2 + x + 1}\,dx$

Here, $(3x - 2)$ can be written as

$$3x - 2 = A\frac{d}{dx}(x^2 + x + 1) + B = A(2x + 1) + B$$

On equating the coefficients of x and constant term from both sides, we get $2A = 3$, $A + B = -2$, then

$$\Rightarrow \quad A = \frac{3}{2} \text{ and } B = \frac{-7}{2} \qquad \textbf{(1)}$$

$$\therefore \quad I = \frac{3}{2}\int (2x + 1)\sqrt{x^2 + x + 1}\,dx - \frac{7}{2}\int \sqrt{x^2 + x + 1}\,dx$$

$$\Rightarrow \quad I = I_1 - I_2 \qquad \text{...(i)}$$

where, $\quad I_1 = \frac{3}{2}\int (2x + 1)\sqrt{x^2 + x + 1}\,dx$

Put $\quad x^2 + x + 1 = t$

$$\Rightarrow \quad 2x + 1 = \frac{dt}{dx}$$

$$\therefore \quad I_1 = \frac{3}{2}\int \sqrt{t}\,dt = \frac{3}{2}\cdot\frac{2}{3}t^{3/2} + C_1$$

$$= t^{3/2} = (x^2 + x + 1)^{3/2} + C_1$$

(1)

and $I_2 = -\frac{7}{2}\int \sqrt{x^2 + x + 1}\,dx$

$$= -\frac{7}{2}\int \sqrt{\left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}}\,dx$$

$$= -\frac{7}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= -\frac{7}{2}\left[\frac{\left(x+\frac{1}{2}\right)}{2}\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 \log\left\{\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right\}\right] + C_2$$

$$= \frac{-7}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x+1} + \frac{3}{8}\log\left\{\frac{2x+1}{2} + \sqrt{x^2+x+1}\right\}\right] + C_2$$

$$= \frac{-7}{8}(2x+1)\sqrt{x^2+x+1} - \frac{21}{16}\log\left|\frac{2x+1}{2} + \sqrt{x^2+x+1}\right| + C_2$$

$$\left[\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + a^2 \log\left|x + \sqrt{x^2+a^2}\right| + C\right] \textbf{(1)}$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I = (x^2+x+1)^{3/2} - \frac{7}{8}(2x+1)\sqrt{x^2+x+1} - \frac{21}{16}\log\left|\frac{(2x+1)}{2} + \sqrt{x^2+x+1}\right| + C$$

where, $C = C_1 + C_2$ **(1)**

39. Find $\int (x+3)\sqrt{3-4x-x^2}\,dx$. **Delhi 2014 C**

Let $I = \int (x+3)\sqrt{3-4x-x^2}dx$

Here, $(x + 3)$ can be written as

$$(x+3) = A \cdot \frac{d}{dx}(3-4x-x^2) + B$$

$$\Rightarrow \quad (x+3) = A(-4-2x) + B \qquad \textbf{(1)}$$

On comparing the coefficients of x and constant term, we get

$$-2A = 1 \text{ and } -4A + B = 3$$

$$\Rightarrow \quad A = \frac{-1}{2} \text{ and } -4 \times \frac{-1}{2} + B = 3$$

$$\Rightarrow \quad A = -\frac{1}{2} \text{ and } B = 1$$

$$\therefore \ (x+3) = -\frac{1}{2}(-4-2x) + 1 \qquad \textbf{(1)}$$

Then, $I = \int \left\{-\frac{1}{2}(-4-2x) + 1\right\}\sqrt{3-4x-x^2}dx$

$$= -\frac{1}{2}\int (-4-2x)\sqrt{3-4x-x^2}dx + \int \sqrt{3-4x-x^2}dx$$

$$= -\frac{1}{2}\int \sqrt{t}\,dt + \int \sqrt{7-(4x+x^2+4)}\,dx$$

(1)

[put $3 - 4x - x^2 = t$]

$$= \left(\frac{-1}{2} \times t^{3/2} \times \frac{2}{3}\right) + \int \sqrt{(\sqrt{7})^2 - (x+2)^2}dx$$

$$= \frac{-1}{3}(3-4x-x^2)^{3/2} + \frac{1}{2}(x+2)\sqrt{3-4x-x^2} + \frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C$$

$$\left[\because \int \sqrt{a^2-x^2}dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot\sin^{-1}\frac{x}{a}\right]$$

(1)

40. Find $\int \frac{5x-2}{1+2x+3x^2}dx.$ **Delhi 2014C; Delhi 2013**

Let $$I=\int \frac{5x-2}{1+2x+3x^2}dx \quad \text{...(i)}$$

Here, $(5x-2)$ can be written as

$$5x-2 = A\frac{d}{dx}(1+2x+3x^2)+B$$

$$\left[\because \text{numerator} = A\cdot\frac{d}{dx}(\text{denominator})+B\right]$$

$\Rightarrow \quad 5x-2 = A(2+6x)+B$

On comparing the coefficients of x and constant terms, we get

$$5=6A \Rightarrow A=\frac{5}{6}$$

and $$-2=2A+B \Rightarrow B=-2A-2$$

$$=-\frac{5}{3}-2=-\frac{11}{3} \quad \left[\because A=\frac{5}{6}\right]$$

Then, from Eq. (i), we get

$$I=\int \frac{\frac{5}{6}(2+6x)}{1+2x+3x^2}dx-\int \frac{\left(\frac{11}{3}\right)}{1+2x+3x^2}dx \quad \textbf{(1)}$$

$$\Rightarrow \quad I=I_1-I_2 \quad \text{...(ii)}$$

where, $$I_1 = \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}dx$$

Put $1+2x+3x^2 = t \Rightarrow (2+6x)dx = dt$

$$\therefore \quad I_1 = \frac{5}{6}\int \frac{dt}{t} = \frac{5}{6}\ln t + C_1$$

$$= \frac{5}{6}\ln|1+2x+3x^2| + C_1 \qquad \textbf{(1)}$$

and $$I_2 = \frac{11}{3}\int \frac{dx}{3x^2+2x+1}$$

$$= \frac{11}{9}\int \frac{dx}{\left(x^2+\frac{2}{3}x+\frac{1}{3}+\frac{1}{9}-\frac{1}{9}\right)}$$

[making a perfect square in denominator]

$$= \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{2}{9}}$$

$$= \frac{11}{9}\cdot\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\left|\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right| + C_2$$

$$\left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{11}{3\sqrt{2}}\tan^{-1}\left|\left(\frac{3x+1}{\sqrt{2}}\right)\right| + C_2 \qquad \textbf{(1)}$$

On putting the values of I_1 and I_2 in Eq. (ii), we get

$$I = \frac{5}{6}\ln|1+2x+3x^2| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

where $C = C_1 + C_2$ **(1)**

41. Find $\int \frac{x^2}{x^4+3x^2+2}dx.$

All India 2014C

Firstly, put $x^2 = t$ and use partial fraction to write integral and in simplest form, then integrate by using suitable formula.

Let $$I = \int \frac{x^2}{x^4+3x^2+2}dx$$

Put $$x^2 = t$$

$\Rightarrow$ $$2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$$ **(1)**

$\therefore$ $$I = \frac{1}{2}\int \frac{t\,dt}{t^2+3t+2}$$

$$= \frac{1}{2}\int \frac{t}{(t+2)(t+1)}dt$$

$$= \frac{1}{2}\left[\int \frac{2}{t+2}dt - \int \frac{-1}{t+1}dt\right]$$ **(1½)**

[by partial fraction]

$$= \frac{1}{2}[2\log|t+2| - \log|t+1|] + C$$

$$= \log\left|\frac{t+2}{\sqrt{t+1}}\right| + C = \log\left|\frac{x^2+2}{\sqrt{x^2+1}}\right| + C$$ **(1½)**

42. Evaluate $\int \frac{x\cos^{-1}x}{\sqrt{1-x^2}}dx.$

All India 2014C; Foreign 2014

Let $\quad I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$

Put $\cos^{-1} x = t$

On differentiating both sides w.r.t. x, we get

$$\frac{-1}{\sqrt{1-x^2}} dx = dt$$

$\therefore \quad I = -\int t \cos t \, dt$ **(1)**

Applying integration by parts, we get

$$I = -\left[t \int \cos t \, dt - \int \left(\frac{d}{dt}(t) \int \cos t \, dt\right) dt\right]$$

$$= -[t \sin t - \int \sin t \, dt] = -t \sin t - \cos t + C$$

(1½)

Substituting the value of t, we get

$$I = -\cos^{-1} x \sqrt{1-\cos^2(\cos^{-1} x)} - \cos(\cos^{-1} x) + C$$

$$\left[\because \sin t = \sqrt{1-\cos^2 t} = \sqrt{1-\cos^2(\cos^{-1} x)}\right]$$

$$\Rightarrow I = -\left(\sqrt{1-x^2}\right) \cos^{-1} x + C$$

(1½)

43. Evaluate $\int \frac{\sin^6 + \cos^6 x}{\sin^2 x \cos^2 x} dx.$ **Delhi 2014C**

Firstly, use $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ to write numerator of integrand in simplest form and then integrate by using suitable method.

Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

$\Rightarrow \quad I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\left[\begin{array}{l}(\sin^2 x + \cos^2 x)^3 \\ -3\sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)\end{array}\right]}{\sin^2 x \cos^2 x} dx \quad \textbf{(1)}$$

$[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)]$

$$= \int \frac{(1)^3 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$[\because \sin^2 x + \cos^2 x = 1]$ **(1½)**

$$= \int \frac{1}{\sin^2 x \cos^2 x} dx - 3\int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx - 3\int 1 dx$$

$$= \int \left[\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x}\right] dx - 3\int 1 dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x) dx - 3\int 1 dx$$

$$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx - 3\int 1 dx$$

$$= \tan x - \cot x - 3x + C$$ **(1½)**

44. Evaluate $\int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x} dx.$ **Delhi 2013C**

Let $I = \int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x} dx$

$= \int \frac{(3\sin x - 2)\cos x}{5 - (1 - \sin^2 x) - 4\sin x} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore \quad I = \int \frac{3t - 2}{5 - (1 - t^2) - 4t} dt$ **(1)**

$= \int \frac{3t - 2}{4 + t^2 - 4t} dt = \int \frac{3t - 2}{(t - 2)^2} dt$

$= \int \frac{3t - 6 + 4}{(t - 2)^2} dt = \int \frac{3(t - 2) + 4}{(t - 2)^2} dt$ **(1)**

$= \int \frac{3}{(t - 2)} dt + \int \frac{4}{(t - 2)^2} dt$

$= 3\log|t - 2| + \frac{4(t - 2)^{-2+1}}{-2 + 1} + C$ **(1)**

$\left[\because \int x^n dx = \frac{x^{n+1}}{n+1}, n \neq -1\right]$

$= 3\log|t - 2| - \frac{4}{(t - 2)} + C$

$= 3\log|\sin x - 2| - \frac{4}{(\sin x - 2)} + C$

[put $t = \sin x$] **(1)**

45. Evaluate $\int e^{2x}\left(\frac{1 - \sin 2x}{1 - \cos 2x}\right) dx.$

Delhi 2013C

Firstly, use trigonometric formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$ to write integral and in simple form, then apply integration by parts to integrate.

Let $I = \int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right) dx$

$$= \int e^{2x}\left(\frac{1-2\sin x\cos x}{2\sin^2 x}\right) dx \qquad \textbf{(1)}$$

$$\left[\because 1-\cos 2x = 2\sin^2 x \text{ and } \sin 2x = 2\sin x\cos x\right]$$

$$= \frac{1}{2}\int e^{2x}(\text{cosec}^2 x - 2\cot x)\, dx \qquad \textbf{(1½)}$$

$$= \frac{1}{2}\int \underset{\text{I}}{e^{2x}}\, \underset{\text{II}}{\text{cosec}^2 x}\, dx - \int e^{2x}\cot x\, dx$$

$$= \frac{1}{2}[-e^{2x}\cot x + \int 2e^{2x}\cot x\, dx] + C - \int e^{2x}\cot x\, dx$$

$$= -\frac{e^{2x}}{2}\cot x + \int e^{2x}\cot x\, dx - \int e^{2x}\cot x\, dx + C$$

$$= -\frac{e^{2x}}{2}\cot x + C \qquad \textbf{(1½)}$$

46. Evaluate $\int \frac{3x+1}{(x+1)^2(x+3)} dx.$ **Delhi 2013C**

Let $I = \int \frac{3x+1}{(x+1)^2(x+3)}\,dx$

Again, let $\frac{3x+1}{(x+1)^2(x+3)}$

$$= \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)} \quad \ldots(i)$$

$$\Rightarrow \quad 3x+1 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

$$\Rightarrow \quad 3x+1 = A(x^2+4x+3) + B(x+3) + C(x^2+1+2x)$$

$$\Rightarrow \quad 3x+1 = (A+C)x^2 + (4A+B+2C)x + 3A+3B+C \quad \textbf{(1)}$$

On comparing like powers of x from both sides, we get

$$A + C = 0$$

$$4A + B + 2C = 3$$

and $\quad 3A + 3B + C = 1$

On solving, we get

$$A = 2, B = -1 \text{ and } C = -2 \quad \textbf{(1)}$$

$\therefore$ Eq. (i) becomes

$$\frac{3x+1}{(x+1)^2(x+3)} = \frac{2}{(x+1)} - \frac{1}{(x+1)^2} - \frac{2}{(x+3)} \quad \textbf{(1)}$$

On integrating both sides, we get

$$I=\int \frac{3x+1}{(x+1)^2(x+3)}dx$$

$$=\int \frac{2}{(x+1)}dx-\int \frac{1}{(x+1)^2}dx-\int \frac{2}{(x+3)}dx$$

$$=2\log|x+1|-\frac{(x+1)^{-2+1}}{(-2+1)}-2\log|x+3|+C$$

$$=2\log\left|\frac{x+1}{x+3}\right|+\frac{1}{(x+1)}+C$$

$$\left[\because \log m-\log n=\log\frac{m}{n}\right]$$ **(1)**

47. Evaluate $\int \frac{\sin(x-a)}{\sin(x+a)}dx.$ **Delhi 2013**

Let $I=\int \frac{\sin(x-a)}{\sin(x+a)}dx$

Put $x+a=t$

$\Rightarrow \quad x=t-a \Rightarrow dx=dt$

$\therefore \quad I=\int \frac{\sin(t-2a)}{\sin t}dt$ **(1)**

$\Rightarrow \quad I=\int \frac{\sin t\cos 2a-\cos t\sin 2a}{\sin t}dt$ **(1)**

$[\because \sin(x-y)=\sin x\cos y-\cos x\sin y]$

$\Rightarrow \quad I=\int \frac{\sin t\cos 2a}{\sin t}dt-\int \frac{\cos t\sin 2a}{\sin t}dt$

$=\cos 2a\int dt-\sin 2a\int \cot t\,dt$ **(1)**

$=t\cos 2a-\sin 2a\ln|\sin t|+C$

$=(x+a)\cos 2a-\sin 2a\ln$

$|\sin(x+a)|+C$ **(1)**

48. Evaluate $\int \frac{x^2}{(x^2+4)(x^2+9)} dx.$

Delhi 2013

Let $I = \int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$

$$= \frac{1}{2}\int \frac{2x^2 + 4 + 9 - 4 - 9}{(x^2 + 4)(x^2 + 9)} dx$$

$$= \frac{1}{2}\int \frac{x^2 + 4}{(x^2 + 4)(x^2 + 9)} dx + \frac{1}{2}\int \frac{x^2 + 9}{(x^2 + 4)(x^2 + 9)} dx$$

$$- \frac{1}{2}\int \frac{13}{(x^2 + 4)(x^2 + 9)} dx \quad \textbf{(1)}$$

$$= \frac{1}{2}\int \frac{dx}{x^2 + 9} + \frac{1}{2}\int \frac{dx}{x^2 + 4}$$

$$- \frac{1}{2}\int \frac{13}{(x^2 + 4)(x^2 + 9)} dx$$

$$= \frac{1}{2} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{2} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right)$$

$$- \frac{1}{2} \cdot \frac{13}{5} \int \left(\frac{1}{(x^2+4)} - \frac{1}{(x^2+9)} \right) dx \ \textbf{(1)}$$

$$\left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} \text{ and apply partial fractions in third term} \right]$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4} \tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{10} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{10} \cdot \frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{x}{3}\right) + \frac{1}{4} \tan^{-1}\left(\frac{x}{2}\right) - \frac{13}{20} \tan^{-1}\left(\frac{x}{2}\right) + \frac{13}{30} \tan^{-1}\left(\frac{x}{3}\right) + C \ \textbf{(1)}$$

$$= \tan^{-1}\left(\frac{x}{3}\right)\left(\frac{1}{6} + \frac{13}{30}\right) + \tan^{-1}\left(\frac{x}{2}\right)\left(\frac{1}{4} - \frac{13}{20}\right) + C$$

$$= \tan^{-1}\left(\frac{x}{3}\right)\left(\frac{5+13}{30}\right) + \tan^{-1}\left(\frac{x}{2}\right)\left(\frac{5-13}{20}\right) + C$$

$$= \frac{18}{30} \tan^{-1}\left(\frac{x}{3}\right) - \frac{8}{20} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{3}{5} \tan^{-1}\left(\frac{x}{3}\right) - \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + C \qquad \textbf{(1)}$$

49. Evaluate $\int \frac{2x^2+1}{x^2(x^2+4)} dx.$

Delhi 2013

Let $I = \int \frac{2x^2+1}{x^2(x^2+4)} dx$

Put $x^2 = t$ and then using partial fraction, we get

$$\frac{2t+1}{t(t+4)} = \frac{A}{t} + \frac{B}{t+4}$$

$\Rightarrow \qquad 2t+1 = A(t+4) + Bt$ **(1/2)**

On comparing the coefficients of t and constant terms, we get

$$2 = A + B \quad \text{and} \quad 1 = 4A \Rightarrow A = \frac{1}{4}$$

$\therefore \qquad B = 2 - A = 2 - \frac{1}{4} = \frac{7}{4}$ **(1)**

Then, $\frac{2x^2+1}{x^2(x^2+4)} = \frac{1}{4x^2} + \frac{7}{4(x^2+4)}$

$\therefore \qquad I = \frac{1}{4}\int \frac{dx}{x^2} + \frac{7}{4}\int \frac{dx}{x^2+4}$ **(1)**

$$= -\frac{1}{4x} + \frac{7}{4} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$= -\frac{1}{4x} + \frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right) + C$ **(1½)**

50. Evaluate $\int \frac{x^2+1}{(x^2+4)(x^2+25)} dx.$

Delhi 2013

Let $I = \int \frac{x^2+1}{(x^2+4)(x^2+25)} dx$

Again, let $\frac{x^2+1}{(x^2+4)(x^2+25)} = \frac{A}{x^2+4} + \frac{B}{x^2+25}$

[by partial fraction] **(1)**

At $x = 0$, $\frac{A}{4} + \frac{B}{25} = \frac{1}{100}$

$\Rightarrow \quad 25A + 4B = 1 \quad$ ...(i) **(1)**

At $x = 1$, $\frac{2}{5 \times 26} = \frac{A}{5} + \frac{B}{26}$

$\Rightarrow \quad \frac{A}{5} + \frac{B}{26} = \frac{1}{65}$

$\Rightarrow \quad 13A + \frac{5}{2}B = 1$

$\Rightarrow \quad 26A + 5B = 2 \quad$ ...(ii) **(1)**

On solving Eqs. (i) and (ii), we get

$A = -\frac{1}{7}, \; B = \frac{8}{7}$

$\therefore \; I = -\frac{1}{7}\int \frac{dx}{x^2+4} + \frac{8}{7}\int \frac{dx}{x^2+25}$

$= -\frac{1}{7} \cdot \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{7} \cdot \frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) + C$

$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$ **(1)**

$= \frac{-1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{8}{35}\tan^{-1}\left(\frac{x}{5}\right) + C$

51. Evaluate $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx.$ **All India 2013**

Let $$I=\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos \alpha}dx$$

$$=\int \frac{(2\cos^2 x-1)-(2\cos^2\alpha-1)}{\cos x-\cos\alpha}dx$$

$[\because \cos 2\theta = 2\cos^2\theta - 1]$ **(1)**

$$=\int \frac{2(\cos^2 x-\cos^2\alpha)}{(\cos x-\cos\alpha)}dx$$

$$=\int \frac{2(\cos x-\cos\alpha)(\cos x+\cos\alpha)}{(\cos x-\cos\alpha)}dx$$

$[\because a^2-b^2=(a+b)(a-b)]$ **(1)**

$$=\int 2(\cos x+\cos\alpha)\,dx$$

$$=2\left[\int \cos x\,dx+\cos\alpha\int dx\right]$$

$\Rightarrow \quad I = 2(\sin x + x\cos\alpha) + C$ **(2)**

52. Evaluate $\int \frac{x+2}{\sqrt{x^2+2x+3}}dx.$ All India 2013

Let $$I=\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$$

$$\Rightarrow \quad I=\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx+\int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$\Rightarrow \quad I=I_1+I_2 \qquad \text{...(i)} \ \mathbf{(1)}$$

where, $$I_1=\int \frac{(x+1)}{\sqrt{x^2+2x+3}}\,dx$$

$$=\int \frac{t}{t}\,dt=\int dt=t+C_1$$

$$\left[\begin{array}{r}\text{put } t^2=x^2+2x+3 \Rightarrow 2t\,dt=(2x+2)dx \\ \Rightarrow t\,dt=(x+1)dx\end{array}\right]$$

$$=\sqrt{x^2+2x+3}+C_1 \qquad \mathbf{(1)}$$

and $$I_2=\int \frac{dx}{\sqrt{x^2+2x+3}}=\int \frac{dx}{\sqrt{(x+1)^2+(\sqrt{2})^2}}$$

$$=\log\{(x+1)+\sqrt{x^2+2x+3}\}+C_2$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}}=\log(x+\sqrt{x^2+a^2})\right] \mathbf{(1)}$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I=\sqrt{x^2+2x+3}+C_1 + \log\{(x+1)+\sqrt{x^2+2x+3}\}+C_2$$

$$=\sqrt{x^2+2x+3} + \log\left\{(x+1)+\sqrt{x^2+2x+3}\right\}+C \ \mathbf{(1)}$$

where, $C=C_1+C_2$

53. Evaluate $\int \frac{dx}{x(x^5+3)}$.

HOTS; All India 2013

Let $$I = \int \frac{dx}{x(x^5+3)} = \int \frac{x^4}{x^5(x^5+3)}\,dx$$

[$\because$ multiplying numerator and denominator by x^4]

Put $t = x^5 \Rightarrow dt = 5x^4\,dx$

$\therefore$ $$I = \int \frac{dt}{5t(t+3)}$$ **(1)**

$$= \frac{1}{5}\int \frac{1}{3}\left[\frac{1}{t} - \frac{1}{t+3}\right]dt$$

$$= \frac{1}{15}[\log|t| - \log|t+3|] + C$$ **(1)**

$$= \frac{1}{15}\log\left|\frac{t}{t+3}\right| + C$$ **(1)**

54. Evaluate $\int \frac{dx}{x(x^3+1)}$. **All India 2013**

15 $|x^5+3|$

Let $\quad I = \int \frac{dx}{x(x^3+1)}$

$$I = \int \frac{dx}{x(x+1)(x^2-x+1)}$$

$[\because a^3 + b^3 = (a+b)(a^2+b^2-ab)]$

Again, let

$$\frac{1}{x(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}$$

$$1 = (x+1)(x^2-x+1)A + Bx(x^2-x+1) + x(x+1)(Cx+D)$$

$$\Rightarrow 1 = (x^3-x^2+x+x^2-x+1)A + B(x^3-x^2+x) + (x^2+x)(Cx+D)$$

$$\Rightarrow 1 = A(x^3+1) + B(x^3-x^2+x) + (Cx^3+Dx^2+Cx^2+Dx)$$

$$\Rightarrow 1 = (A+B+C)x^3 + (-B+D+C)x^2 + (B+D)x + A \quad \textbf{(1)}$$

On comparing the coefficients of different powers of x from both sides, we get

$$A+B+C=0 \quad \text{...(i)}$$

$$-B+D+C=0 \quad \text{...(ii)}$$

$$B+D=0 \quad \text{...(iii)}$$

and $\quad A=1 \quad$ ...(iv)

From Eqs. (ii) and (iii), we get

$$C-2B=0 \quad \text{...(v)}$$

From Eqs. (i) and (iv), we get

$$B+C=-1 \quad \text{...(vi)}$$

From Eqs. (v) and (vi), we get

$$B = -\frac{1}{3},\ D = \frac{1}{3} \text{ and } C = -\frac{2}{3} \qquad \textbf{(1)}$$

$$\therefore I = \int \frac{dx}{x(x^3+1)} = \int \left[\frac{1}{x} - \frac{1}{3(x+1)} + \frac{-\frac{2x}{3} + \frac{1}{3}}{x^2 - x + 1} \right] dx$$

$$\Rightarrow I = \int \frac{dx}{x} - \frac{1}{3} \int \frac{dx}{x+1} + \frac{1}{3} \int \frac{1-2x}{x^2 - x + 1} dx$$

$$\Rightarrow I = \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\int \frac{2x-1}{x^2-x+1} dx \quad \textbf{(1)}$$

Put $\qquad t = x^2 - x + 1$

$\Rightarrow \qquad dt = (2x-1)dx$

$$\therefore \quad I = \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\int \frac{dt}{t}$$

$$= \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\log|t| + C$$

$$= \log|x| - \frac{1}{3}\log|x+1| - \frac{1}{3}\log|x^2 - x + 1| + C$$

$[\text{put } t = x^2 - x + 1]$

$$= \log|x| - \frac{1}{3}\log|(x+1)(x^2 - x + 1)| + C$$

$[\because \log m + \log n = \log mn]$

$$= \log|x| - \frac{1}{3}\log|x^3 + 1| + C$$

$$= \log|x| - \log|x^3+1|^{1/3} + C = \log\frac{|x|}{|x^3+1|^{1/3}} + C$$

$\left[\because \log m - \log n = \log\frac{m}{n}\right]$ **(1)**

55. Evaluate $\int \frac{dx}{x(x^3+8)}$. **All India 2013**

Let $I = \int \frac{dx}{x(x^3+8)} = \int \frac{dx}{x(x^3+2^3)}$

$$= \int \frac{dx}{x(x+2)(x^2-2x+4)} \quad \text{...(i)}$$

$[\because x^3 + a^3 = (x+a)(x^2 + a^2 - ax)]$

Now, for partial fraction method, write

$$\frac{1}{x(x+2)(x^2-2x+4)} = \frac{A}{x} + \frac{B}{x+2} + \frac{Cx+D}{x^2-2x+4}$$

$$\Rightarrow \quad 1 = A(x+2)(x^2-2x+4) + Bx(x^2-2x+4) + (Cx+D)(x^2+2x)$$

$$\Rightarrow \quad 1 = A(x^3-2x^2+4x+2x^2-4x+8) + B(x^3-2x^2+4x) + (Cx^3+2Cx^2+Dx^2+2Dx)$$

$$\Rightarrow \quad 1 = A(x^3+8) + B(x^3-2x^2+4x) + (Cx^3+2Cx^2+Dx^2+2Dx) \; \textbf{(1)}$$

$$\Rightarrow \quad 1 = (A+B+C)x^3 + (-2B+2C+D)x^2 + (4B+2D)x + 8A$$

On comparing the coefficients of different powers of x from both sides, we get

$$A+B+C=0 \quad \text{...(ii)}$$

$$-2B+2C+D=0 \quad \text{...(iii)}$$

$$4B+2D=0 \quad \text{...(iv)}$$

and $\quad 8A = 1$

$\therefore \quad A = \frac{1}{8}$

From Eqs. (iii) and (iv), we get

$$-2B+2C-2B=0 \Rightarrow -4B+2C=0$$

$\therefore \quad C = 2B \quad \text{...(v)}$

On putting the values of C and A in Eq. (ii), we get

$$\frac{1}{8} + B + 2B = 0$$

$$\Rightarrow \quad B = -\frac{1}{24} \text{ and } C = -\frac{1}{12}$$

1

and $$D = \frac{1}{12} \quad \textbf{(1)}$$

Now, substituting the values of A, B, C and D in the given integral, we get

$$\frac{1}{x(x+2)(x^2-2x+4)} = \frac{1}{8x} - \frac{1}{24(x+2)} + \frac{\frac{-1}{12}x + \frac{1}{12}}{x^2-2x+4}$$

$$\therefore \quad I = \int \left[\frac{1}{8x} - \frac{1}{24(x+2)} + \frac{\left(\frac{-x}{12} + \frac{1}{12} \right)}{x^2-2x+4} \right] dx$$

$$= \frac{1}{8}\int \frac{dx}{x} - \frac{1}{24}\int \frac{dx}{x+2} - \frac{1}{12}\int \frac{x-1}{x^2-2x+4}\,dx \quad \textbf{(1)}$$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|x+2| - \frac{1}{24}\int \frac{2x-2}{x^2-2x+4}\,dx$$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|x+2|$$

$$- \frac{1}{24}\log|x^2 - 2x + 4| + C$$

$[\text{let } x^2 - 2x + 4 = t \Rightarrow (2x-2)dx = dt]$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|(x+2)(x^2 - 2x + 4)| + C$$

$[\because \log m + \log n = \log mn]$

$$= \frac{1}{8}\log|x| - \frac{1}{24}\log|(x^3 + 8)| + C$$

$[\because (a+b)(a^2 - ab + b^2) = a^3 + b^3]$

$$= \frac{1}{8}\left\{\log|x| - \frac{1}{3}\log|x^3 + 8|\right\} + C$$

$$= \frac{1}{8}\{\log|x| - \log|x^3 + 8|^{1/3}\} + C$$

$$= \frac{1}{8}\log\left|\frac{x}{(x^3+8)^{1/3}}\right| + C$$ **(1)**

56. Evaluate $\int \sin x \cdot \sin 2x \cdot \sin 3x\,dx$ **HOTS; Delhi 2012**

? It is a product of three trigonometric functions. So, firstly we take two functions at a time and use the relation $2\sin A \sin B = \cos(A-B) - \cos(A+B)$ and then integrate it.

Let $I = \int \sin x \sin 2x \sin 3x \, dx$

$$= \frac{1}{2}\int \sin x \,(2\sin 2x \sin 3x)\, dx$$

[multiplying numerator and denominator by 2]

$$= \frac{1}{2}\int \sin x [\cos(2x-3x) - \cos(2x+3x)]\, dx$$

$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$ **(1)**

$$= \frac{1}{2}\int \sin x [\cos(-x) - \cos 5x]\, dx$$

$$= \frac{1}{2}\int \sin x (\cos x - \cos 5x)\, dx \quad [\because \cos(-x) = \cos x]$$

$$= \frac{1}{2}\int \sin x \cos x \, dx - \frac{1}{2}\int \sin x \cos 5x \, dx \qquad \textbf{(1)}$$

$$= \frac{1}{4}\int 2\sin x \cos x \, dx - \frac{1}{4}\int (2\sin x \cos 5x)\, dx$$

[multiplying numerator and denominator by 2]

$$= \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int \{\sin(x+5x)$$

$$\left[\begin{array}{l}\because 2\sin x \cos x = \sin 2x \text{ and} \\ 2\sin A \cos B = \sin(A+B) + \sin(A-B)\end{array}\right]$$

$$= \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int [\sin 6x + \sin(-4x)]\, dx \qquad \textbf{(1)}$$

$$= \frac{1}{4}\int \sin 2x \, dx - \frac{1}{4}\int (\sin 6x - \sin 4x)\, dx$$

$[\because \sin(-\theta) = -\sin\theta]$

$$= \frac{-1}{4}\cdot\frac{\cos 2x}{2} - \frac{1}{4}\left[\frac{-\cos 6x}{6} + \frac{\cos 4x}{4}\right] + C$$

$$\left[\because \int \sin ax \, dx = \frac{-\cos ax}{a}\right]$$

$$= \frac{-\cos 2x}{8} + \frac{\cos 6x}{24} - \frac{\cos 4x}{16} + C \qquad \textbf{(1)}$$

57. Evaluate $\int \frac{2}{(1-x)(1+x^2)} dx.$ **Delhi 2012**

> Here, denominator is a product of two algebraic functions. So, firstly we use partial fraction method and then integrate it.

Let $$I = \int \frac{2}{(1-x)(1+x^2)} dx$$

Again, let $$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} \quad \ldots(i)$$

(1)

$$\Rightarrow \frac{2}{(1-x)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$$

$$\Rightarrow 2 = A(1+x^2)+(Bx+C)(1-x)$$

$$\Rightarrow 2 = A + Ax^2 + Bx + C - Bx^2 - Cx$$

$$\Rightarrow 2 = (A-B)x^2 + (B-C)x + (A+C)$$

(1)

On comparing coefficients of x^2, x and constant terms from both sides, we get

$$A - B = 0 \quad \ldots(ii)$$

$$B - C = 0 \quad \ldots(iii)$$

and $$A + C = 2 \quad \ldots(iv)$$

On adding Eqs. (ii) and (iii), we get

$$A - C = 0 \quad \ldots(v)$$

On adding Eqs. (iv) and (v), we get

$$2A = 2 \Rightarrow A = 1$$

On putting $A = 1$ in Eq. (ii), we get $B = 1$.

On putting $B = 1$ in Eq. (iii), we get $C = 1$.

Hence, $A = 1, B = 1$ and $C = 1$ **(1)**

Now, Eq. (i) becomes

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2}$$

On integrating both sides w.r.t. x, we get

$$\int \frac{2}{(1-x)(1+x^2)} dx = \int \frac{1}{1-x} dx + \int \frac{x+1}{1+x^2} dx$$

$$= -\log|1-x| + \int \frac{x}{1+x^2} dx + \int \frac{1}{1+x^2} dx$$

$$= -\log|1-x| + \frac{1}{2}\log|1+x^2| + \tan^{-1} x + C$$

$$\left[\begin{array}{l} \text{put } 1+x^2 = t \\ \Rightarrow \ 2x\,dx = dt \Rightarrow x\,dx = dt/2 \\ \therefore \int \frac{x}{1+x^2} dx = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| = \frac{1}{2}\log|1+x^2| \end{array}\right]$$

(1)

58. Evaluate $\int \left(\frac{1+\sin x}{1+\cos x}\right) e^x dx.$ All India 2012C

Let $I = \int \left(\frac{1+\sin x}{1+\cos x}\right) e^x \, dx$

$$= \int \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} \cdot e^x \, dx \quad \textbf{(1)}$$

$$\left[\because \sin m = 2\sin\frac{m}{2}\cos\frac{m}{2} \text{ and } 1+\cos m = 2\cos^2\frac{m}{2}\right]$$

$$= \int \left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right) e^x \, dx \quad \textbf{(1)}$$

$$= \int e^x \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right) dx \quad \textbf{(1)}$$

On comparing it with

$\int e^x [f(x) + f'(x)] dx = e^x f(x)$, we get

$$f(x) = \tan\frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}\sec^2\frac{x}{2}$$

$$\therefore \quad I = e^x \tan\frac{x}{2} + C \quad \textbf{(1)}$$

59. Evaluate $\int \frac{x^2}{(x\sin x + \cos x)^2} dx$. **All India 2012C**

Let $\quad I = \int \frac{x^2}{(x\sin x + \cos x)^2} dx$

$\Rightarrow \quad I = \frac{x\cos x}{(x\sin x + \cos x)^2} \cdot x\sec x \, dx \quad$ ...(i)**(1)**

Again, let $\quad x\sin x + \cos x = t$

$\Rightarrow \quad (x\cos x + \sin x - \sin x)\, dx = dt$

$\Rightarrow \quad x\cos x \, dx = dt$

$\therefore \quad I_1 = \int \frac{x\cos x \, dx}{(x\sin x + \cos x)^2}$

$$= \int \frac{dt}{t^2} = \frac{-1}{t} = \frac{-1}{x\sin x + \cos x} \quad \textbf{(1)}$$

Now, integrating Eq. (i) by parts, we get

$$I = \int \underset{\text{I}}{x \sec x} \cdot \underset{\text{II}}{\frac{x \cos x}{(x \sin x + \cos x)^2}}\, dx$$

$$= x \sec x \cdot \frac{(-1)}{x \sin x + \cos x}$$

$$- \int (1 \cdot \sec x + x \sec x \tan x) \cdot \frac{-dx}{x \sin x + \cos x} \quad \textbf{(1)}$$

$$= \frac{-x \sec x}{x \sin x + \cos x} + \int \sec x \left(1 + \frac{x \sin x}{\cos x}\right) \frac{dx}{x \sin x + \cos x}$$

$$= \frac{-x \sec x}{x \sin x + \cos x} + \int \sec^2 x\, dx$$

$$= \frac{-x \sec x}{x \sin x + \cos x} + \tan x + C \quad \textbf{(1)}$$

60. Evaluate $\int e^{2x} \sin x\, dx$. **Foreign 2011**

Let $I = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\sin x}\, dx$

On taking $\sin x$ as I function and e^{2x} as II function and integrating by parts, we get

$$I = \sin x \int e^{2x}\, dx - \int \left\{ \frac{d}{dx}(\sin x) \int e^{2x} dx \right\} dx \quad \textbf{(1)}$$

$$\Rightarrow \quad I = \frac{\sin x \cdot e^{2x}}{2} - \int \frac{\cos x \cdot e^{2x}}{2}\, dx$$

$$\Rightarrow \quad I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x\, dx$$

$$\Rightarrow \quad I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} I_1 \quad \ldots\text{(i)} \; \textbf{(1)}$$

where, $I_1 = \int \underset{\text{II}}{e^{2x}} \underset{\text{I}}{\cos x}\, dx$

On integrating by parts again by taking $\cos x$ as I function and e^{2x} as II function, we get

$$I_1 = \cos x \int e^{2x}\, dx - \int \left\{ \frac{d}{dx}(\cos x) \int e^{2x} dx \right\} dx$$

$$\Rightarrow \quad I_1 = \frac{\cos x \cdot e^{2x}}{2} - \int \frac{(-\sin x) \cdot e^{2x}}{2}\, dx$$

$$\Rightarrow \quad I_1 = \frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x\, dx$$

$$\Rightarrow \quad I_1 = \frac{e^{2x} \cos x}{2} + \frac{1}{2} I \quad \ldots\text{(ii)}$$

$$\left[\because \int e^{2x} \sin x\, dx = I \right] \; \textbf{(1)}$$

On putting the value of I_1 from Eq. (ii) in Eq. (i), we get

$$I = \frac{e^{2x}\sin x}{2} - \frac{1}{2}\left[\frac{e^{2x}\cos x}{2} + \frac{1}{2}I\right]$$

$$\Rightarrow \quad I = \frac{e^{2x}\sin x}{2} - \frac{e^{2x}\cos x}{4} - \frac{1}{4}I$$

$$\Rightarrow \quad I + \frac{1}{4}I = e^{2x}\left(\frac{\sin x}{2} - \frac{\cos x}{4}\right)$$

$$\Rightarrow \quad \frac{5I}{4} = e^{2x}\left(\frac{2\sin x - \cos x}{4}\right)$$

$$\Rightarrow \quad I = \frac{4}{5}e^{2x}\left(\frac{2\sin x - \cos x}{4}\right)$$

$$\Rightarrow \quad I = 1/5\, e^{2x}(2\sin x - \cos x) \qquad \textbf{(1)}$$

61. Evaluate $\int \frac{3x+5}{\sqrt{x^2-8x+7}}\,dx$.

Foreign 2011

Let $$I = \int \frac{3x+5}{\sqrt{x^2-8x+7}}\,dx$$

Here, $(3x + 5)$ can be written as

$$3x + 5 = A \cdot \frac{d}{dx}(x^2 - 8x + 7) + B$$

$$\Rightarrow \quad 3x + 5 = A(2x - 8) + B \quad \text{...(i)}$$

$$\Rightarrow \quad 3x + 5 = 2Ax + (B - 8A) \quad \textbf{(1/2)}$$

On comparing the coefficients of x and constant terms from both sides, we get

$$2A = 3 \quad \text{...(ii)}$$

and $$-8A + B = 5 \quad \text{...(iii)}$$

From Eq. (ii), we get $A = \frac{3}{2}$

On putting $A = \frac{3}{2}$ in Eq. (iii), we get

$$-8\left(\frac{3}{2}\right) + B = 5$$

$$\Rightarrow \quad -12 + B = 5$$

$$\Rightarrow \quad B = 17 \quad \textbf{(1/2)}$$

On putting the values of A and B in Eq. (i), we get

$$3x + 5 = \frac{3}{2}(2x - 8) + 17 \quad \text{...(iv)}$$

Hence, the given integral can be written as

$$I = \int \frac{\frac{3}{2}(2x-8)+17}{\sqrt{x^2-8x+7}}\,dx \quad \text{[using Eq. (iv)]}$$

$$\Rightarrow \quad I = \frac{3}{2}\int \frac{2x-8}{\sqrt{x^2-8x+7}}\,dx + 17\int \frac{dx}{\sqrt{x^2-8x+7}}$$

$$\Rightarrow \quad I = \frac{3}{2} I_1 + 17 I_2 \quad \text{...(v)} \ \textbf{(1)}$$

$$\int \frac{2x-8}{}$$

where, $I_1 = \int \frac{2x-8}{\sqrt{x^2-8x+7}}\,dx$

Put $x^2 - 8x + 7 = t$

$\Rightarrow \quad (2x-8)\,dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt$

$\Rightarrow \quad I_1 = \frac{t^{1/2}}{1/2}$

$\Rightarrow \quad I_1 = 2t^{1/2} \Rightarrow I_1 = 2\sqrt{t}$ **(1/2)**

$\Rightarrow \quad I_1 = 2\sqrt{x^2-8x+7}$

and $I_2 = \int \frac{dx}{\sqrt{x^2-8x+7}}$

$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{x^2-8x+7+16-16}}$

$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{(x-4)^2-9}}$

$= \int \frac{dx}{\sqrt{(x-4)^2-(3)^2}}$

$\therefore \quad I_2 = \log\left|(x-4) + \sqrt{(x-4)^2-(3)^2}\right|$

$\left[\because \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x + \sqrt{x^2-a^2}\right|\right]$ **(1)**

Hence, putting the values of I_1 and I_2 in Eq. (v), we get

$$I = \frac{3}{2}\left(2\sqrt{x^2-8x+7}\right) + 17\log\left|(x-4) + \sqrt{(x-4)^2-9}\right| + C$$

$$\Rightarrow \quad I = 3\sqrt{x^2-8x+7} + 17\log\left|(x-4) + \sqrt{(x-4)^2-9}\right| + C$$ **(1/2)**

62. Evaluate $\int \frac{x^2+4}{x^4+16}\,dx.$ **All India 2011C**

Firstly, divide numerator and denominator by x^2 and reduce the integrand in standard form.

Let $$I = \int \frac{x^2+4}{x^4+16}\,dx$$

On dividing numerator and denominator by x^2, we get

$$I = \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x^2+\frac{16}{x^2}\right)}dx = \int \frac{\left(1+\frac{4}{x^2}\right)}{\left(x-\frac{4}{x}\right)^2+8}dx \quad \textbf{(1)}$$

Put $$x-\frac{4}{x}=t \Rightarrow \left(1+\frac{4}{x^2}\right)dx = dt$$

$$\therefore \quad I = \int \frac{dt}{t^2+8} = \int \frac{dt}{t^2+(2\sqrt{2})^2} \quad \textbf{(1)}$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right)+C$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{4}{x}}{2\sqrt{2}}\right)+C$$

$$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2\sqrt{2}x}\right)+C \quad \textbf{(2)}$$

63. Evaluate $\int \frac{x^2+1}{x^4+1}\,dx.$ **Delhi 2011C**

Let $I = \int \frac{x^2+1}{x^4+1}\,dx$

On dividing numerator and denominator by x^2, we get

$$I = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)}\,dx = \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}+2-2\right)}\,dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+(\sqrt{2})^2}\,dx \qquad \textbf{(1½)}$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ **(1)**

$$\therefore \quad I = \int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$\left[\because \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C \qquad \textbf{(1)}$$

$$I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + C \qquad \textbf{(1/2)}$$

64. Evaluate $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}\,dx.$

HOTS; Delhi 2011C

In this type of integral, first we make a term in denominator such that whose differential coefficient present in numerator and then integrate it.

Let $I = \int \frac{\sin x - \cos x}{\sqrt{\sin 2x}} dx$

$$= \int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x - 1}} dx \qquad \textbf{(1)}$$

$$= \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2 \sin x \cos x - 1}} dx$$

$$[\because \sin^2 \theta + \cos^2 \theta = 1]$$

$$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 1}} dx \qquad \textbf{(1)}$$

Put $\sin x + \cos x = t$

$\Rightarrow$ $(\cos x - \sin x)\, dx = dt$

$\therefore$ $I = \int \frac{-dt}{\sqrt{t^2 - 1}} = -\log (t + \sqrt{t^2 - 1}) + C$

$$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log (x + \sqrt{x^2 - a^2})\right] \textbf{(1)}$$

$\Rightarrow$ $I = -\log |(\sin x + \cos x)$

$$+ \sqrt{(\sin x + \cos x)^2 - 1}| + C$$

$$[\because t = \sin x + \cos x]$$

$$= -\log |(\sin x + \cos x) + \sqrt{\sin 2x}| + C \quad \textbf{(1)}$$

65. Evaluate $\int \frac{2x}{(x^2+1)(x^2+3)}dx.$ **Delhi 2011**

Let $I = \int \frac{2x}{(1+x^2)(x^2+3)}dx$

Put $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$ **(1)**

$\therefore \quad I = \int \frac{dt}{(t+1)(3+t)}$

Again, let $\frac{1}{(t+1)(3+t)} = \frac{A}{(1+t)} + \frac{B}{3+t}$ ...(i)

(1)

$\Rightarrow \quad 1 = A(3+t) + B(1+t)$

On putting $t=-3$, we get

$$1=-2B \Rightarrow B=-\frac{1}{2}$$

Now, on putting $t=-1$, we get

$$1=2A \quad \Rightarrow \quad A=1/2$$

On putting $A=\frac{1}{2}$ and $B=-\frac{1}{2}$ in Eq. (i), we get

$$\frac{1}{(1+t)(3+t)}=\frac{1/2}{1+t}+\frac{-1/2}{3+t}$$ **(1/2)**

On integrating both sides, we get

$$\int \frac{1}{(1+t)(3+t)}dt=\frac{1}{2}\int \frac{1}{1+t}dt-\frac{1}{2}\int \frac{1}{3+t}dt$$

$$=\frac{1}{2}\log|1+t|-\frac{1}{2}\log|3+t|$$

$$\left[\because \int \frac{dx}{x}=\log|x|\right]$$

$$=\frac{1}{2}\log|1+x^2|-\frac{1}{2}\log|3+x^2|+C$$

$$[\because t=x^2]$$

$$\therefore \qquad I=\frac{1}{2}\log\left|\frac{1+x^2}{3+x^2}\right|+C$$

$$\left[\because \log m-\log n=\log\frac{m}{n}\right]$$ **(1½)**

66. Evaluate $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx.$

Delhi 2011; All India 2010

Do same as Que 37.

[**Ans.** $5\sqrt{x^2+4x+10}$

$-7\log|x+2+\sqrt{x^2+4x+10}|+C$]

67. Evaluate $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx.$ **All India 2010C**

Do same as Que 45. $\left[\textbf{Ans. } \frac{1}{2} e^{2x} \tan x + C\right]$

68. Evaluate $\int \frac{dx}{(x^2+1)(x^2+2)}$. **Delhi 2010C**

Do same as Que 65.

$$\left[\textbf{Ans. } \tan^{-1} x - \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C\right]$$

69. Evaluate $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$.

HOTS; Delhi 2010C

Use integration by parts i.e.

$\int u \cdot v\, dx = \left[u \int v dx - \int \left\{\frac{d}{dx} u \int v dx\right\} dx\right]$ and also

remember ILATE whenever use it.

Let $$I = \int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$$

$$= \int \underset{\text{I}}{\log(\log x)} \cdot \underset{\text{II}}{1}\, dx + \int \frac{1}{(\log x)^2} dx \quad \textbf{(1/2)}$$

Using integration by parts in first integral, we get

$$I = \log(\log x)\int 1dx - \int\left[\frac{d}{dx}\log(\log x)\int 1dx\right]dx$$
$$+\int \frac{1}{(\log x)^2}dx + C \quad \textbf{(1/2)}$$

$$= \log(\log x)\cdot x - \int \frac{1}{\log x}\cdot\frac{1}{x}\cdot x\,dx$$
$$+\int \frac{1}{(\log x)^2}dx + C$$

$$= x\log(\log x) - \int \underset{\text{I}}{(\log x)^{-1}}\ \underset{\text{II}}{1}\,dx$$
$$+\int \frac{1}{(\log x)^2}dx + C \quad \textbf{(1)}$$

Again, applying integration by parts in the middle integral, we get

$$I = x\log(\log x) - [(\log x)^{-1}\int 1dx$$
$$-\int\left\{\frac{d}{dx}(\log x)^{-1}\int 1dx\right\}dx\Bigg] + \int\frac{1}{(\log x)^2}dx + C$$

(1)

$$= x\log(\log x) - \left[\frac{x}{\log x} - \int -(\log x)^{-2}\cdot\frac{1}{x}\cdot x\,dx\right]$$
$$+\int \frac{1}{(\log x)^2}dx + C$$

$$= x\log(\log x) - \frac{x}{\log x} - \int\frac{1}{(\log x)^2}dx$$
$$+\int \frac{1}{(\log x)^2}dx + C$$

$$= x\log(\log x) - \frac{x}{\log x} + C \quad \textbf{(1)}$$

70. Evaluate $\int \frac{x+2}{\sqrt{(x-2)(x-3)}}\,dx.$ **All India 2010**

Do same as Que 61.

$$\left[\textbf{Ans. } \sqrt{x^2-5x+6}+\frac{9}{2}\log\left|\left(x-\frac{5}{2}\right)+\sqrt{(x-2)(x-3)}\right|+C\right]$$

71. Evaluate $\int \frac{1-x^2}{x(1-2x)}\,dx.$ **Delhi 2010**

Let $$I=\int \frac{1-x^2}{x(1-2x)}\,dx=\int \frac{1-x^2}{x-2x^2}\,dx$$

Given integral can be rewritten as

$$I=\int \left[\frac{1}{2}+\frac{1-\frac{1}{2}x}{x(1-2x)}\right]dx \qquad \textbf{(1)}$$

$$\Rightarrow \qquad I = \frac{1}{2}\int dx + \int \frac{1-\frac{1}{2}x}{x(1-2x)}\,dx \quad \text{...(i)}$$

Let $$\frac{\left(1-\frac{1}{2}x\right)}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x} \quad \text{...(ii)}$$

$$\Rightarrow \qquad 1-\frac{1}{2}x = A(1-2x) + Bx \quad \text{...(iii)} \ \textbf{(1)}$$

On putting $x = 0$ and $x = \frac{1}{2}$ in Eq. (iii), we get

$$1 - 0 = A(1-0) + 0$$

$$\Rightarrow \qquad A = 1$$

and $$1-\frac{1}{2}\left(\frac{1}{2}\right) = A\left[1-2\left(\frac{1}{2}\right)\right] + B\left(\frac{1}{2}\right)$$

$$\Rightarrow \qquad 1-\frac{1}{4} = A(1-1) + \frac{1}{2}B$$

$$\Rightarrow \qquad \frac{3}{4} = \frac{1}{2}B$$

$$\Rightarrow \qquad B = \frac{3}{2} \quad \textbf{(1)}$$

On putting the values of A and B in Eq (ii), we get

$$\frac{1-\frac{1}{2}x}{x(1-2x)} = \frac{1}{x} + \frac{3/2}{1-2x}$$

Then, from Eq. (i), we get

$$I = \frac{1}{2}\int dx + \int \frac{1}{x}\,dx + \int \frac{3/2}{1-2x}\,dx$$

$$= \frac{1}{2}x + \log|x| + \frac{3}{2}\,\frac{\log|1-2x|}{-2} + C$$

$$\left[\because \int \frac{1}{a-x}\,dx = -\frac{1}{a}\log|a-x|\right]$$

$$= \frac{1}{2}x + \log|x| - \frac{3}{4}\log|1-2x| + C \quad \textbf{(1)}$$

72. Evaluate $\int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x}\right) dx.$ **Delhi 2010**

Let $I = \int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x}\right) dx$

$$= \int e^x \left(\frac{2 \sin 2x \cos 2x - 4}{2 \sin^2 2x}\right) dx \qquad \textbf{(1)}$$

$$= \int e^x \left(\frac{2 \sin 2x \cos 2x}{2 \sin^2 2x} - \frac{4}{2 \sin^2 2x}\right) dx \qquad \textbf{(1)}$$

$$= \int e^x (\cot 2x - 2 \operatorname{cosec}^2 2x)\, dx$$

On comparing with

$\int e^x[f(x) + f'(x)]\, dx = e^x f(x) + C$, we get

$f(x) = \cot 2x$

$\Rightarrow f'(x) = -2 \operatorname{cosec}^2 2x$

$\therefore \quad I = e^x \cot 2x + C$ **(2)**

73. Evaluate $\int \frac{(\sin x + \cos x)}{\sqrt{\sin 2x}} dx.$ **HOTS; All India 2009C**

Do same as Ques 64.

[**Ans.** $\sin^{-1}(\sin x - \cos x) + C$]

use $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$

74. Evaluate $\int \frac{2x + 5}{\sqrt{7 - 6x - x^2}} dx.$ **Delhi 2009C**

Let $I = \int \frac{2x+5}{\sqrt{7-6x-x^2}}\,dx$

Given integral can be written as

$$I = \int \frac{-(-2x-6)-1}{\sqrt{7-6x-x^2}}\,dx$$

$$= -\int \frac{-2x-6}{\sqrt{7-6x-x^2}}\,dx - \int \frac{dx}{\sqrt{7-6x-x^2}}$$

$\Rightarrow \quad I = -I_1 - I_2$ ...(i) **(1)**

where, $I_1 = \int \frac{-2x-6}{\sqrt{7-6x-x^2}}\,dx$

Put $7-6x-x^2 = t$

$\Rightarrow \quad (-6-2x)\,dx = dt$

$\therefore$

$$I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt$$

$$= 2\sqrt{t} = 2\sqrt{7-6x-x^2}$$ **(1)**

And $I_2 = \int \frac{dx}{\sqrt{7-6x-x^2}}$

$$= \int \frac{dx}{\sqrt{-(-7+6x+x^2+9-9)}}$$

$$= \int \frac{dx}{\sqrt{-[(x+3)^2-16]}}$$

$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{(4)^2-(x+3)^2}}$

$$= \sin^{-1}\left(\frac{x+3}{4}\right) + C$$ **(1)**

$$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}\right]$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I = -2\sqrt{7-6x-x^2} - \sin^{-1}\left(\frac{x+3}{4}\right) + C \quad \textbf{(1)}$$

75. Evaluate $\int \frac{dx}{\sqrt{5-4x-2x^2}}$.

All India 2009

Firstly, make a perfect square in denominator part and then integrate it using suitable formula.

Let $I = \int \frac{dx}{\sqrt{5-4x-2x^2}}$

$$= \int \frac{dx}{\sqrt{-2\left(x^2+2x-\frac{5}{2}\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left(x^2+2x-\frac{5}{2}+1-1\right)}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x^2+2x+1)-\frac{5}{2}-1\right]}} \quad \textbf{(1)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x+1)^2-\left(\frac{5}{2}+1\right)\right]}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{-\left[(x+1)^2-\frac{7}{2}\right]}} \quad \textbf{(1)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\sqrt{\frac{7}{2}}\right)^2-(x+1)^2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \sin^{-1}\left(\frac{x+1}{\sqrt{\frac{7}{2}}}\right) + C \quad \textbf{(1)}$$

$\left(\sqrt{2} \right)$

$$\left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \right]$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left[\frac{\sqrt{2}(x+1)}{\sqrt{7}} \right] + C \qquad \textbf{(1)}$$

76. Evaluate $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx.$ **HOTS; Delhi 2009**

Firstly, put $e^x = t \Rightarrow e^x dx = dt$, Then, do same as Que 75.

$$\left[\textbf{Ans. } \sin^{-1} \left(\frac{e^x + 2}{3} \right) + C \right]$$

77. Evaluate $\int \frac{x+3}{x^2 - 2x - 5} dx.$ **All India 2008C**

Let $\quad I = \int \frac{x+3}{x^2 - 2x - 5} dx$

Here $(x + 3)$ can be written as

$$x + 3 = A + B \frac{d}{dx} (x^2 - 2x - 5)$$

$\Rightarrow \qquad x + 3 = A + B(2x - 2) \qquad$ **(1)**

On equating the coefficients of like terms from both sides, we get

$$2B = 1 \Rightarrow B = 1/2$$

and $\qquad A - 2B = 3$

$\Rightarrow \qquad A - 2 \times \frac{1}{2} = 3$

$\Rightarrow \qquad A = 3 + 1 = 4 \qquad$ **(1)**

$\therefore \qquad I = \int \frac{4 + \frac{1}{2}(2x - 2)}{x^2 - 2x - 5} dx$

$\Rightarrow \qquad I = 4 \int \frac{1}{x^2 - 2x - 5} dx$

$$+ \frac{1}{2} \int \frac{2x - 2}{x^2 - 2x - 5} dx \qquad \textbf{(1)}$$

On putting $x^2 - 2x - 5 = t \Rightarrow (2x-2)\,dx = dt$ in second integral, we get

$$I = 4\int \frac{dx}{(x-1)^2 - 1 - 5} + \frac{1}{2}\int \frac{dt}{t}$$

$$= 4\int \frac{dx}{(x-1)^2 - (\sqrt{6})^2} + \frac{1}{2}\log|t| + C$$

$$= 4 \cdot \frac{1}{2\sqrt{6}} \log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + \frac{1}{2}\log|x^2 - 2x - 5| + C$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{2}{\sqrt{6}} \log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + \frac{1}{2}\log|x^2 - 2x - 5| + C \quad \textbf{(1)}$$

78. Evaluate $\int x \sin^{-1} x\, dx$. **All India 2008C**

Use integration by parts

$$\int \underset{\text{I}}{u} \cdot \underset{\text{II}}{v}\, dx = \left[u \int v\, dx - \int \left\{\frac{d}{dx} u \int v\, dx\right\} dx\right]$$

Let $$I = \int x \sin^{-1} x \, dx$$

Taking x as 1st function and $\sin^{-1} x$ as 2nd function and using the rule of integration by parts, we get

$$I = \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} dx + C$$

$$= \frac{x^2}{2} \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx + C \qquad \textbf{(1)}$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2-1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^2}{2} \sin^{-1} x + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} dx - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2} \left[x^2 \sin^{-1} x + \int \sqrt{1-x^2}\, dx - \int \frac{dx}{\sqrt{1-x^2}} \right] \qquad \textbf{(1)}$$

$$\Rightarrow \quad I = \frac{1}{2} \left[x^2 \sin^{-1} x + \frac{x}{2} \cdot \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x - \sin^{-1} x \right] \qquad \textbf{(1)}$$

$$\left[\because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \right.$$

$$\left. \text{and} \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= \frac{1}{2} \left[x^2 \sin^{-1} x + \frac{x}{2} \cdot \sqrt{1-x^2} - \frac{1}{2} \sin^{-1} x \right] + C$$

$$\therefore \quad I = \frac{x^2}{2} \sin^{-1} x + \frac{x}{4} \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + C \qquad \textbf{(1)}$$

79. Evaluate $\int x \cdot \log |(x+1)| dx$. **Delhi 2008C**

Let $I = \int \underset{II}{x} \cdot \underset{I}{\log|(x+1)|}\, dx$

Using integration by parts, we get

$$I = \log|(x+1)| \int x\,dx - \int \left[\frac{d}{dx}\log|(x+1)| \int x\,dx\right] dx \quad \textbf{(1)}$$

$$= \log|(x+1)| \cdot \frac{x^2}{2} - \int \frac{1}{x+1} \cdot \frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2}\log|(x+1)| - \frac{1}{2}\int \frac{x^2}{x+1}\,dx$$

$$= \frac{x^2}{2}\log|(x+1)| - \frac{1}{2}\int \left(x - 1 + \frac{1}{x+1}\right) dx \quad \textbf{(1½)}$$

$$\left[\begin{array}{r|l} & x-1 \\ x+1 & x^2 \\ & x^2 + x \\ & \underline{-\quad -} \\ & -x \\ & -x-1 \\ & \underline{+\quad +} \\ & 1 \end{array}\right]$$

$$= \frac{x^2}{2}\log|(x+1)| - \frac{1}{2}\left[\frac{x^2}{2} - x + \log|(x+1)|\right] + C \quad \left[\because \int \frac{dx}{x} = \log|x|\right]$$

$$\Rightarrow I = \frac{x^2}{2}\log|(x+1)| - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\log|(x+1)| + C$$

$$\therefore I = \frac{1}{2}|(x^2-1)|\log|(x+1)| - \frac{x^2}{4} + \frac{x}{2} + C \quad \textbf{(1½)}$$

6 Mark Questions

80. Evaluate $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}\,dx.$

All India 2014

Firstly, divide numerator and denominator by $\cos^4 x$ to convert integrand in terms of $\tan x$ and then put $\tan x = t$ and convert integrand into standard form which can integrate easily.

Let $I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$

On dividing numerator and denominator by $\cos^4 x$ in RHS, we get

$$I = \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx \quad \textbf{(1)}$$

$$= \int \frac{(\sec^2 x)(\sec^2 x)}{\tan^4 x + \tan^2 x + 1} dx \quad \textbf{(1)}$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$ and

$\sec^2 x = 1 + \tan^2 x = 1 + t^2$ **(1)**

$$\therefore \quad I = \int \frac{1+t^2}{t^4 + t^2 + 1} dt \Rightarrow I = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 1} dt$$

$$\Rightarrow \quad I = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 3} dt \quad \textbf{(1)}$$

Again, put $u = t - \frac{1}{t} \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = du$

$$\therefore \quad I = \int \frac{du}{u^2 + (\sqrt{3})^2}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C \qquad \textbf{(1)}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{3}}\right) + C \qquad \left[\because u = t - \frac{1}{t}\right]$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{3}t}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{3} \tan x}\right) + C \; [\because t = \tan x]$$

(1)

81. Evaluate $\int (\sqrt{\cot x} + \sqrt{\tan x})\, dx$.

All India 2014; Delhi 2010 C

(1)

Let $I = \int [\sqrt{\cot x} + \sqrt{\tan x}]\, dx$

$= \int \sqrt{\tan x}\,(1 + \cot x)\, dx$

Put $\tan x = t^2 \Rightarrow \sec^2 x\, dx = 2t\, dt$

or $\quad dx = \dfrac{2t}{1+t^4}$ **(1)**

$[\because 1 + \tan^2 x = \sec^2 x \Rightarrow 1 + t^4 = \sec^2 x]$

$$\therefore \quad I = \int t\left(1 + \frac{1}{t^2}\right)\frac{2t}{(1+t^4)}\, dt$$

$$\Rightarrow \quad I = 2\int \frac{t^2+1}{t^4+1}\, dt \quad \textbf{(1)}$$

On dividing numerator and denominator by t^2 in RHS, we get

$$I = 2\int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}\, dt = 2\int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\, dt \quad \textbf{(1)}$$

Again, put $t - \dfrac{1}{t} = y \Rightarrow \left(1 + \dfrac{1}{t^2}\right) dt = dy$

$$\therefore \quad I = 2\int \frac{dy}{y^2 + (\sqrt{2})^2}$$

$$\Rightarrow \quad I = \sqrt{2}\tan^{-1}\frac{y}{\sqrt{2}} + C \quad \textbf{(1)}$$

$$= \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + C \quad \left[\because y = t - \frac{1}{t}\right] \textbf{(1)}$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + C \quad [\because t^2 = \tan x]$$

(1)

82. Evaluate $\int \frac{1}{\cos^4 x + \sin^4 x} dx$. **All India 2014**

Let $I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$

On dividing numerator and denominator by $\cos^4 x$ in RHS, we get

$$I = \int \frac{\sec^4 x}{1 + \tan^4 x} dx \Rightarrow I = \int \frac{(\sec^2 x)(\sec^2 x)}{1 + (\tan^2 x)^2} dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x (1 + \tan^2 x)}{1 + (\tan^2 x)^2} dx \quad \textbf{(1)}$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$\therefore \quad I = \int \frac{1 + t^2}{1 + t^4} dt \quad \textbf{(1)}$$

Again, dividing numerator and denominator by t^2 in RHS, we get

$$I = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} + 2 - 2} dt = \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt \quad \textbf{(1)}$$

Put $t - \frac{1}{t} = u$

$$\Rightarrow \quad \left(1 + \frac{1}{t^2}\right) dt = du \quad \textbf{(1)}$$

Then, $I = \int \frac{du}{u^2 + (\sqrt{2})^2}$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + C \quad \left[\because u = t - \frac{1}{t}\right] \textbf{(1)}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2} \tan x}\right) + C \quad [\because t = \tan x]$$

(1)

83. Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx.$ **Delhi 2014C**

Firstly, put $x^2 = t$ and apply partial fraction to convert integrand into some standard form which can integrate easily.

$$\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx$$

Let $x^2 = t$, then

$$\therefore \quad \frac{t}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4} \quad \text{...(i)}$$

$$\Rightarrow \quad t = A(t+4) + B(t+1)$$

$$\Rightarrow \quad t = (A+B)t + 4A + B \quad \textbf{(1)}$$

On comparing the coefficients of like powers from both sides, we get

$$A + B = 1,\ 4A + B = 0 \quad \textbf{(1)}$$

On solving these equations, we get

$$A = -\frac{1}{3} \text{ and } B = \frac{4}{3} \quad \textbf{(1)}$$

From Eq. (i), we get

$$\frac{t}{(t+1)(t+4)} = \frac{-\frac{1}{3}}{t+1} + \frac{\frac{4}{3}}{t+4} \quad \textbf{(1)}$$

$$\therefore \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = -\frac{1}{3}\int \frac{1}{x^2+1}\,dx + \frac{4}{3}\int \frac{1}{x^2+4}\,dx \quad [\because x^2 = t] \quad \textbf{(1)}$$

$$= -\frac{1}{3}\tan^{-1} x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

$$\left[\because \int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}\right] \quad \textbf{(1)}$$

84. Find $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx, x \in [0, 1]$.

All India 2014C

Firstly, use the identity $\sin^{-1}\theta + \cos^{-1}\theta = \frac{\pi}{2}$ to convert integrand in terms of $\sin^{-1}$ only. Then, integrate by using substitution.

Let $I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}$

We know that, $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \pi/2$

$$\Rightarrow \quad \cos^{-1}\sqrt{x} = \frac{\pi}{2} - \sin^{-1}\sqrt{x}$$

$$\therefore \qquad I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\pi/2} dx$$

$$\therefore \int \frac{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}}{\frac{\pi}{2}} dx = \frac{2}{\pi}\int \left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}dx - \int 1dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}dx - x$$

$$\Rightarrow \qquad I = \frac{4}{\pi} I_1 - x + C \qquad \text{...(i)} \ \mathbf{(1)}$$

where, $I_1 = \int \sin^{-1}\sqrt{x}\, dx$

Put $\sqrt{x} = t\ z \Rightarrow x = t^2$ and $dx = 2t\, dt$

$$I_1 = \int \sin^{-1} t\, 2t\, dt = 2\int \sin^{-1} t \cdot t\, dt$$

$$= 2\left[\sin^{-1} t \cdot \frac{t^2}{2} - \int \frac{1}{\sqrt{1-t^2}} \cdot \frac{t^2}{2} dt\right] \ \mathbf{(1)}$$

[using integration by parts]

$$= \int \frac{t^2}{\sqrt{1-t^2}} dt = t^2 \sin^{-1} t - \int \frac{(1-t^2)+1}{\sqrt{1-t^2}} dt$$

$$= t^2 \sin^{-1} t + \int \sqrt{1-t^2}\, dt - \int \frac{1}{\sqrt{1-t^2}}\, dt \quad \mathbf{(1)}$$

$$= t^2 \sin^{-1} t + \frac{t\sqrt{1-t^2}}{2} + \frac{1}{2}\sin^{-1} t - \sin^{-1} t$$

$$\left(t^2 - \frac{1}{2}\right) \sin^{-1} t + \frac{1}{2} r \sqrt{1-t^2} \ \mathbf{(1)}$$

$$= \frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}]$$

$$= \frac{1}{2}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] \qquad \mathbf{(1)}$$

On putting the value of I_1 in Eq. (i), we get

$$\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} dx$$

$$= \frac{2}{\pi}[(2x-1)\sin^{-1}\sqrt{x} + \sqrt{x-x^2}] - x + C \ \mathbf{(1)}$$

85. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx.$ **Delhi 2014C**

$$\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$$

The integrand $\frac{x^2+x+1}{(x+1)^2(x+2)}$ is a proper rational function.

$$\therefore \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \quad ...(i)$$

(1)

$$\Rightarrow x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

$$\Rightarrow x^2+x+1 = A(x^2+3x+2) + B(x+2) + C(x^2+2x+1)^2$$

$$\Rightarrow x^2+x+1 = (A+C)x^2 + (3A+B+2C)x + (2A+2B+C)$$ **(1)**

On comparing the coefficients of like powers from both sides, we get **(1)**

$A+C=1, 3A+B+2C=1$ and $2A+2B+C=1$

On solving these equations, we get **(1)**

$$A=-2, B=1 \text{ and } C=3$$

From Eq. (i), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$ **(1)**

$$\therefore \int \frac{x^2+x+1}{(x+1)^2(x+2)}dx = -2\int \frac{1}{x+1}dx + \int \frac{dx}{(x+1)^2} + 3\int \frac{dx}{(x+2)}$$

$$= -2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + C$$ **(1)**

86. Find $\int \frac{\sqrt{x^2+1}(\log(x^2+1)-2\log x)}{x^4}dx.$

All India 2014C

Let $= I\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$

$$= \int \frac{\sqrt{x^2+1}\log\left(\frac{x^2+1}{x^2}\right)}{x^4}dx$$

$\left[\because \log m - a\log n = \log\frac{m}{n^a}\right]$ **(1)**

$$= \int \frac{x\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)}{x^4}dx \quad \textbf{(1)}$$

$$= \int \frac{\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)}{x^3}dx$$

Put $1+\frac{1}{x^2} = t$, then $\frac{-2}{x^3}dx = dt \Rightarrow \frac{dx}{x^3} = -\frac{dt}{2}$ **(1)**

$$\therefore \quad I = -\frac{1}{2}\int \sqrt{t}\log t\,dt$$

$$= -\frac{1}{2}\left[\log t \times \frac{t^{3/2}}{3/2} - \int \frac{t^{3/2}}{3/2}\times\frac{1}{t}dt\right]$$

[using integration by parts] **(1)**

$$= -\frac{1}{3}[t^{3/2}\log t - \int \sqrt{t}dt]$$

$$= -\frac{1}{3}\left[t^{3/2}\log t - \frac{t^{3/2}}{3/2}\right] + C \quad \textbf{(1)}$$

$$= -\frac{1}{3}t^{3/2}\left[\log t - \frac{2}{3}\right] + C$$

$$= -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

$\left[\because t = 1+\frac{1}{x^2}\right]$ **(1)**

87. Evaluate $\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}dx.$ **Delhi 2012**

> Firstly, put $x = \sin t$ and then use integration by parts and simplify it.

Let $$I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$$

Put $\sin^{-1} x = t \Rightarrow x = \sin t$

$\Rightarrow$ $$\frac{1}{\sqrt{1-x^2}} dx = dt \qquad \textbf{(1½)}$$

$\therefore$ $$I = \int \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\, dt$$

Using integration by parts, taking t as the first function and $\sin t$ as the second function, we get

$$I = t \int \sin t\, dt - \int \left[\frac{d}{dt}(t) \cdot \int \sin t\, dt\right] dt \quad \textbf{(1½)}$$

$$\Rightarrow \quad I = -t \cos t - \int (1 \times - \cos t)\, dt$$

$$= -t \cos t + \int \cos t\, dt$$

$$\Rightarrow \quad I = -t \cos t + \sin t + C \qquad \textbf{(1½)}$$

$$\Rightarrow \quad I = -t\sqrt{1-\sin^2 t} + \sin t + C$$

$$[\because \cos^2 t = 1 - \sin^2 t \Rightarrow \cos t = \sqrt{1-\sin^2 t}]$$

$$\therefore \quad I = -\sin^{-1} x \sqrt{1-x^2} + x + C$$

$$[\because t = \sin^{-1} x \text{ and } x = \sin t] \quad \textbf{(1½)}$$

88. Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}dx.$ **Delhi 2012**

Using partial fraction, such that

$$\frac{1}{(x-a)^2(x+b)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x+b)}$$

and then integrate it to get the desired result.

Let $$I = \int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$$

Again, let

$$\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} \quad \text{...(i)} \ \mathbf{(1)}$$

$$\Rightarrow \frac{x^2+1}{(x-1)^2(x+3)}$$

$$= \frac{A(x-1)(x+3) + B(x+3) + C(x-1)^2}{(x-1)^2(x+3)}$$

$$\Rightarrow x^2+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

$$\Rightarrow x^2+1 = A(x^2+2x-3) + B(x+3)$$
$$+ C(x^2+1-2x)$$

$$\Rightarrow x^2+1 = (A+C)x^2 + (2A+B-2C)x$$
$$+ (-3A+3B+C)$$

On comparing the coefficients of x^2, x and constant terms from both sides, we get

$$A + C = 1 \quad \text{...(ii)}$$

$$2A + B - 2C = 0 \quad \text{...(iii)}$$

$$-3A + 3B + C = 1 \quad \text{...(iv)} \ \mathbf{(1)}$$

On multiplying Eq. (iii) by 3 and subtracting it from Eq. (iv), we get

$$-9A + 7C = 1 \quad \text{...(v)}$$

On multiplying Eq. (ii) by 7 and subtracting it from Eq. (v), we get

$$-16A = -6 \quad \text{...(vi)}$$

$$\therefore \quad A = \frac{6}{16} = \frac{3}{8} \quad \mathbf{(1)}$$

On putting $A = \frac{3}{8}$ in Eq. (ii), we get

$$\frac{3}{8} + C = 1 \Rightarrow C = 1 - \frac{3}{8} = \frac{5}{8}$$

$$\frac{3}{8}+C=1 \Rightarrow C=1-\frac{3}{8}=\frac{5}{8}$$

On putting $A=\frac{3}{8}$ and $C=\frac{5}{8}$ in Eq. (iii), we get

$$\frac{3}{4}+B-\frac{5}{4}=0 \Rightarrow B-\frac{2}{4}=0 \Rightarrow B=\frac{2}{4}=\frac{1}{2}$$

Thus, $A=\frac{3}{8}, B=\frac{1}{2}$ and $C=\frac{5}{8}$ **(1)**

$\therefore$ Eq. (i) becomes

$$\frac{x^2+1}{(x-1)^2(x+3)}=\frac{3/8}{x-1}+\frac{1/2}{(x-1)^2}+\frac{5/8}{x+3}$$

On integrating both sides, we get

$$I=\int\frac{x^2+1}{(x-1)^2(x+3)}dx=\frac{3}{8}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{(x-1)^2}+\frac{5}{8}\int\frac{dx}{x+3}$$ **(1)**

$$=\frac{3}{8}\log|x-1|+\frac{1}{2}\left(\frac{-1}{x-1}\right)+\frac{5}{8}\log|x+3|+C$$

Hence, $I=\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$ **(1)**

89. Evaluate $\int\frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$. **All India 2011**

> If the integral is of the form $\int\frac{gx+d}{\sqrt{ax^2+bx+c}}dx$, then we take $gx+d=A\cdot\frac{d}{dx}(ax^2+bx+c)+B$ and then integrate it.

Let $I=\int\frac{(6x+7)\,dx}{\sqrt{(x-5)(x-4)}} \Rightarrow I=\int\frac{(6x+7)\,dx}{\sqrt{x^2-9x+20}}$

Here, $(6x+7)$ can be written as

$$6x+7=A\cdot\frac{d}{dx}(x^2-9x+20)+B$$

$\Rightarrow$ $6x+7=A(2x-9)+B$ ...(i) **(1)**

$\Rightarrow \quad 6x + 7 = A(2x - 9) + B$ ...(i) **(1)**

On comparing the coefficients of x and constant terms from both sides, we get

$$2A = 6 \Rightarrow A = 3 \text{ and } -9A + B = 7$$

$$\Rightarrow \quad -9(3) + B = 7 \qquad [\because A = 3]$$

$$\Rightarrow \quad -27 + B = 7$$

$$\therefore \quad B = 34$$ **(1)**

On putting the values of A and B in Eq. (i), we get $\quad 6x + 7 = 3(2x - 9) + 34$

$\therefore$ Given integral can be written as

$$I = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}}\, dx$$

$$\Rightarrow I = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}\, dx + 34\int \frac{dx}{\sqrt{x^2-9x+20}}$$

$$\Rightarrow I = 3I_1 + 34I_2$$ ...(ii) **(1)**

where, $I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}}\, dx$

Put $\quad x^2 - 9x + 20 = t \Rightarrow (2x-9)\, dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = 2t^{1/2} = 2\sqrt{t}$$

$$= 2\sqrt{x^2-9x+20}$$ ...(iii) **(1)**

and $\quad I_2 = \int \frac{dx}{\sqrt{x^2-9x+20}}$

$$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{x^2-9x+20+\frac{81}{4}-\frac{81}{4}}}$$

$$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 + \left(20 - \frac{81}{4}\right)}}$$

$$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}$$

$$\Rightarrow \quad I_2 = \int \frac{dx}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$\therefore I_2 = \log\left|\left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| \quad \ldots \text{(iv)}$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}|\right] \quad \textbf{(1)}$$

On putting the values of I_1 and I_2 from Eqs. (iii) and (iv) in Eq. (ii), we get

$$I = 3(2\sqrt{x^2 - 9x + 20})$$

$$+ 34\left[\log\left|\left(x - \frac{9}{2}\right) + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}\right|\right] + C$$

$$\therefore \quad I = 6\sqrt{x^2 - 9x + 20}$$

$$+ 34\log\left|x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}\right| + C \quad \textbf{(1)}$$

90. Evaluate $\int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx.$ **All India 2009C**

Firstly, use the method of partial fraction and then integrate it.

Let $$I = \int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx$$

Again, let $$\frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1} \quad \textbf{(1)}$$

$$= \frac{A(x^2 + 1) + (Bx + C)(x + 2)}{(x + 2)(x^2 + 1)}$$

$$\Rightarrow x^2 + x + 1 = A(x^2 + 1) + (Bx + C)(x + 2) \quad \text{...(i)}$$

$$\Rightarrow x^2 + x + 1 = (A + B)x^2 + (C + 2B)x + (A + 2C)$$

On putting $x = -2$ in Eq. (i), we get

$4 - 2 + 1 = A(4 + 1) + 0 \Rightarrow 3 = 5A \Rightarrow A = 3/5$

On equating the coefficients of x^2 and constant terms, we get

$$A + B = 1 \quad \text{...(ii)}$$

$$A + 2C = 1 \quad \text{...(iii)}$$

On putting $A = \frac{3}{5}$ in Eq. (ii), we get

$$\frac{3}{5} + B = 1 \Rightarrow B = 1 - \frac{3}{5} = \frac{2}{5}$$

On putting $A = \frac{3}{5}$ in Eq. (iii), we get

$$\frac{3}{5} + 2C = 1 \Rightarrow 2C = 1 - \frac{3}{5}$$

$$\Rightarrow \quad 2C = 2/5 \Rightarrow C = 1/5 \quad \textbf{(2)}$$

$$\therefore \quad I = \int \frac{3/5}{x+2} dx + \int \frac{\frac{2}{5}x + \frac{1}{5}}{x^2+1} dx \quad \textbf{(1)}$$

$$= \frac{3}{5}\int \frac{dx}{x+2} + \frac{2}{5}\int \frac{xdx}{x^2+1} + \frac{1}{5}\int \frac{dx}{x^2+1}$$

Put $x^2 + 1 = t \Rightarrow 2x\,dx = dt$

$$\therefore \quad I = \frac{3}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{dt}{t} + \frac{1}{5}\int \frac{dx}{(x)^2 + (1)^2} \quad \textbf{(1)}$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log|t| + \frac{1}{5}\cdot\frac{1}{1}\tan^{-1}\frac{x}{1} + C$$

$$\left[\because \int \frac{dx}{x} = \log|x| \text{ and } \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{3}{5}\log|x+2| + \frac{1}{5}\log|x^2+1| + \frac{1}{5}\tan^{-1}x + C \quad \textbf{(1)}$$

91. Evaluate $\int \frac{\tan x + \tan^3 x}{1 + \tan^3 x} dx.$

HOTS; Delhi 2009C

Firstly, simplify the given integrand in initial level and then apply the method of partial fraction after that integrate it and get the desired result.

Let $I = \int \frac{\tan x + \tan^3 x}{1+\tan^3 x} dx$

$$= \int \frac{\tan x(1+\tan^2 x)}{1+\tan^3 x} dx = \int \frac{\tan x \sec^2 x}{1+\tan^3 x} dx$$

$[\because 1+\tan^2 x = \sec^2 x]$ **(1)**

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore \quad I = \int \frac{t\, dt}{1+t^3}$

$[\because (a^3+b^3) = (a+b)(a^2-ab+b^2)]$

$$= \int \frac{t\, dt}{(1+t)(1+t^2-t)} \quad \textbf{(1)}$$

Let $\quad \frac{t}{(1+t)(1+t^2-t)} = \frac{A}{1+t} + \frac{Bt+C}{t^2-t+1}$

$\Rightarrow \quad t = A(t^2-t+1) + (Bt+C)(1+t)$

$\Rightarrow \quad t = (A+B)t^2 + (-A+B+C)t + (A+C)$

On putting $t=-1$, we get

$-1 = A(1+1+1)+0$

$\Rightarrow \quad A = \frac{-1}{3}$

On equating the coefficients of t^2 and constant terms from both sides, we get $A+B=0$

$\Rightarrow \quad \frac{-1}{3} + B = 0 \Rightarrow B = \frac{1}{3}$ and $A+C=0$

$\Rightarrow \quad -\frac{1}{3} + C = 0 \Rightarrow C = \frac{1}{3}$ **(1)**

$\therefore \quad I = \int \frac{(-1/3)}{1+t} dt + \int \frac{\frac{1}{3}t + \frac{1}{3}}{t^2-t+1} dt$

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{3}\int \frac{t+1}{t^2-t+1} dt$$

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t-1+3}{t^2-t+1}\,dt$$

$$= -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{2t-1}{t^2-t+1}\,dt + \frac{3}{6}\int \frac{dt}{t^2-t+1+\frac{1}{4}-\frac{1}{4}} \quad \textbf{(1)}$$

Let $z = t^2 - t + 1$

$\Rightarrow dz = (2t-1)\,dt$ in middle integral, we get

$$I = -\frac{1}{3}\int \frac{dt}{1+t} + \frac{1}{6}\int \frac{1}{z}\,dz + \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)+\frac{3}{4}}$$

$$\Rightarrow I = -\frac{1}{3}\log|1+t| + \frac{1}{6}\log z + \frac{1}{2}\int \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad \textbf{(1)}$$

$$= -\frac{1}{3}\log|1+\tan x| + \frac{1}{6}\log(t^2-t+1) + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\sqrt{3}/2}\right) + C \quad [\because t = \tan x]$$

$$= -\frac{1}{3}\log|1+\tan x| + \frac{1}{6}\log\left|\tan^2 x - \tan x + 1\right| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x - 1}{\sqrt{3}}\right) + C \quad \textbf{(1)}$$

92. Evaluate $\int \frac{x^2}{x^4+x^2+1}\,dx.$ **HOTS; Delhi 2008C**

Let $I = \int \frac{x^2}{x^4+x^2+1}\,dx = \frac{1}{2}\int \frac{2x^2}{x^4+x^2+1}\,dx$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{x^2+1+x^2-1}{x^4+x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{x^2+1}{x^4+x^2+1}\,dx + \frac{1}{2}\int \frac{x^2-1}{x^4+x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2} I_1 + \frac{1}{2} I_2 \qquad \ldots\text{(i)} \ \mathbf{(1)}$$

where $\quad I_1 = \int \frac{x^2+1}{x^4+x^2+1} dx$

On dividing numerator and denominator by x^2, we get

$$I_1 = \int \frac{\left(\frac{x^2}{x^2} + \frac{1}{x^2}\right)}{\left(\frac{x^4}{x^2} + \frac{x^2}{x^2} + \frac{1}{x^2}\right)} dx \qquad \mathbf{(1)}$$

$$\Rightarrow \quad I_1 = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x^2 + 1 + \frac{1}{x^2}\right)} dx$$

$$\Rightarrow \quad I_1 = \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 2 + 1} dx$$

$$\Rightarrow \quad I_1 = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + (\sqrt{3})^2} dx \qquad \mathbf{(1)}$$

Put $\quad x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{t^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{t}{\sqrt{3}} \qquad \left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{\left(x - \frac{1}{x}\right)}{\sqrt{3}} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{x\sqrt{3}}\right) \qquad \mathbf{(1)}$$

and $\quad I_2 = \int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$

$x^4 + x^2 + 1$

On dividing numerator and denominator by x^2, we get

$$I = \int \frac{\left(\frac{x^2}{x^2} - \frac{1}{x^2}\right)}{\left(\frac{x^4}{x^2} + \frac{x^2}{x^2} + \frac{1}{x^2}\right)} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x^2 + 1 + \frac{1}{x^2}\right)} dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2 + 1} dx = \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 1} dx$$

Put $x + \frac{1}{x} = t \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$

$$\therefore \quad I_2 = \int \frac{dt}{t^2 - 1} = \int \frac{dt}{t^2 - (1)^2}$$

$$\left[\because \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{2 \times 1} \log \left| \frac{t-1}{t+1} \right| \qquad \textbf{(1)}$$

$$= \frac{1}{2} \log \left| \frac{\left(x + \frac{1}{x}\right) - 1}{\left(x + \frac{1}{x}\right) + 1} \right| \quad \left[\because t = x + \frac{1}{x}\right]$$

$$= \frac{1}{2} \log \left| \frac{x^2 + 1 - x}{x^2 + 1 + x} \right|$$

Now, on putting the values of I_1 and I_2 in Eq.(i), we get

$$I = \frac{1}{2} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{x\sqrt{3}} \right)$$

$$+ \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{x^2 - x + 1}{x^2 + x + 1} \right| + C$$

$$\therefore \quad I = \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x^2 - 1}{x\sqrt{3}}\right) + \frac{1}{4} \log\left|\frac{x^2 - x + 1}{x^2 + x + 1}\right| + C \quad \textbf{(1)}$$

Note In this type of integral, we cannot use the method of partial fraction directly.