

Integrals

1.

$\int_a^b f(x) dx$ is equal to :

(2024)

(A) $\int_a^b f(a-x) dx$

(B) $\int_a^b f(a+b-x) dx$

(C) $\int_a^b f(x-(a+b)) dx$

(D) $\int_a^b f((a-x) + (b-x)) dx$

Ans.

(B) $\int_a^b f(a+b-x) dx$

2. Find : (2024)

$$\int x \sqrt{1+2x} dx$$

Ans. $1+2x = t^2$

$$2 dx = 2t dt$$

$$\begin{aligned} \frac{1}{2} \int (t^4 - t^2) dt &= \frac{1}{2} \left[\frac{t^5}{5} - \frac{t^3}{3} \right] + C \\ &= \frac{(1+2x)^{\frac{5}{2}}}{10} - \frac{(1+2x)^{\frac{3}{2}}}{6} + C \end{aligned}$$

3. Evaluate : (2024)

$$\int_0^{\frac{\pi}{4}} \frac{\sin \sqrt{x}}{\sqrt{x}} dx$$

Ans.

$$\int_0^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}} dx \quad \text{Put } \sqrt{x} = t \Rightarrow dx = 2t dt$$

$$2 \int_0^{\frac{\pi}{2}} \sin t dt = 2 [-\cos t]_0^{\frac{\pi}{2}}$$

$$= 2$$

4. Find : (2024)

$$\int \frac{x^2}{(x^2 + 4)(x^2 + 9)} dx$$

Ans.

$$\text{Let } I = \int \frac{x^2}{(x^2+4)(x^2+9)} dx$$

$$\text{Put } x^2 = t$$

$$\frac{t}{(t+4)(t+9)} = \frac{A}{t+4} + \frac{B}{t+9} \Rightarrow A = \frac{-4}{5}, B = \frac{9}{5}$$

$$I = \frac{-4}{5} \int \frac{1}{2^2+x^2} dx + \frac{9}{5} \int \frac{1}{3^2+x^2} dx$$

$$= \frac{-2}{5} \tan^{-1}\left(\frac{x}{2}\right) + \frac{3}{5} \tan^{-1}\left(\frac{x}{3}\right) + C$$

5. Evaluate : (2024)

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx$$

Ans.

$$\int_1^3 (|x-1| + |x-2| + |x-3|) dx$$

$$= \int_1^3 (x-1) dx + \int_1^2 -(x-2) dx + \int_2^3 (x-2) dx - \int_1^3 (x-3) dx$$

$$= \int_1^3 2 dx + \int_1^2 (2-x) dx + \int_2^3 (x-2) dx$$

$$= [2x]_1^3 + \left[\frac{(2-x)^2}{-2}\right]_1^2 + \left[\frac{(x-2)^2}{2}\right]_2^3$$

$$= 4 + \frac{1}{2} + \frac{1}{2} = 5$$

6. Evaluate : (2024)

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Ans.

$$\text{Let } I = \int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$$

Put $\sin x - \cos x = t$, so that $(\cos x + \sin x) dx = dt$

$$\sin^2 x + \cos^2 x - \sin 2x = t^2 \Rightarrow \sin 2x = 1 - t^2$$

$$I = \int_{-1}^0 \frac{dt}{25 - 16t^2}$$

$$= \frac{1}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$$

$$= \frac{1}{40} \left[\log \left| \frac{5+4t}{5-4t} \right| \right]_{-1}^0$$

$$= \frac{1}{40} \left[\log 1 - \log \left(\frac{1}{9} \right) \right] = \frac{1}{40} \log 9 \text{ or } \frac{1}{20} \log 3$$

7. Evaluate : (2024)

$$\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Ans.

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$$

Put $\sin x = t$ so that $\cos x dx = dt$

$$I = 2 \int_0^1 t \tan^{-1} t dt$$

$$= 2 \left[\tan^{-1} t \left(\frac{t^2}{2} \right) - \frac{1}{2} \int \frac{t^2}{1+t^2} dt \right]_0^1$$

$$= 2 \left[\left(\frac{t^2}{2} \right) \tan^{-1} t - \frac{1}{2} t + \frac{1}{2} \tan^{-1} t \right]_0^1$$

$$= 2 \left(\frac{\pi}{4} - \frac{1}{2} \right) = \frac{\pi}{2} - 1$$

Previous Years' CBSE Board Questions

7.2 Integration as an Inverse Process of Differentiation

MCQ

1. $\int \frac{\sec x}{\sec x - \tan x} dx$ equals
- (a) $\sec x - \tan x + c$ (b) $\sec x + \tan x + c$
 (c) $\tan x - \sec x + c$ (d) $-(\sec x + \tan x) + c$ (2023)

VSA (1 mark)

2. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$ (AI 2017)
3. Write the antiderivative of $\left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ (Delhi 2014)
4. Evaluate: $\int \cos^{-1}(\sin x) dx$ (Delhi 2014)
5. Evaluate: $\int \frac{dx}{\sin^2 x \cos^2 x}$ (Foreign 2014, Delhi 2014 C) (U)
6. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$ (Delhi 2014 C)

SA I (2 marks)

7. Find: $\int \sqrt{1 - \sin 2x} dx, \frac{\pi}{4} < x < \frac{\pi}{2}$ (Delhi 2019)
8. Evaluate: $\int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} dx$ (2018) (Ap)

7.3 Methods of Integration

MCQ

9. $\int e^{5 \log x} dx$ is equal to
- (a) $\frac{x^5}{5} + C$ (b) $\frac{x^6}{6} + C$
 (c) $5x^4 + C$ (d) $6x^5 + C$ (2023)
10. $\int x^2 e^{x^3} dx$ equals
- (a) $\frac{1}{3} e^{x^3} + C$ (b) $\frac{1}{3} e^{x^4} + C$
 (c) $\frac{1}{2} e^{x^3} + C$ (d) $\frac{1}{2} e^{x^2} + C$ (2020) (Ap)
11. $\int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$ is equal to
- (a) $\tan(xe^x) + c$ (b) $\cot(xe^x) + c$
 (c) $\cot(e^x) + c$ (d) $\tan[e^x(1+x)] + c$ (2020) (Ap)

VSA (1 mark)

12. Find: $\int \frac{2x}{\sqrt[3]{x^2+1}} dx$ (2020)
13. Find: $\int \frac{\sin^6 x}{\cos^8 x} dx$ (AI 2014 C)

SA I (2 marks)

14. Find: $\int \frac{dx}{\sqrt{4x-x^2}}$ (Term II, 2021-22) (Ev)

SA II (3 marks)

15. Find: $\int \frac{\sin x}{\sin(x-2a)} dx$ (Term II, 2021-22 C)
16. Find: $\int \frac{1}{e^x+1} dx$ (Term II, 2021-22)
17. Find: $\int \frac{2x}{(x^2+1)(x^2+2)} dx$ (Term II, 2021-22)

LA I (4 marks)

18. Integrate the function $\frac{\cos(x+a)}{\sin(x+b)}$ with respect to x . (AI 2019)
19. Find $\int \frac{(3\sin\theta-2)\cos\theta}{5-\cos^2\theta-4\sin\theta} d\theta$. (Delhi 2016)
20. Evaluate: $\int \frac{\sin(x-a)}{\sin(x+a)} dx$ (Foreign 2015)
21. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$ (Delhi 2014)

7.4 Integrals of Some Particular Functions

VSA (1 mark)

22. Find: $\int \frac{dx}{\sqrt{9-4x^2}}$ (2020)
23. Find: $\int \frac{dx}{9+4x^2}$ (2020)

SA I (2 marks)

24. Find: $\int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$ (Delhi 2019) (Ap)
25. Find: $\int \frac{dx}{\sqrt{5-4x-2x^2}}$ (AI 2019)
26. Find: $\int \frac{dx}{x^2+4x+8}$ (Delhi 2017)
27. Find: $\int \frac{dx}{5-8x-x^2}$ (AI 2017)

SA II (3 marks)

28. Find: $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$ (2023)

LA I (4 marks)

29. Find $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}} dx$. (Delhi 2016) (Ap)

30. Find $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx$. (AI 2015, 2014) (Ap)

31. Evaluate: $\int \frac{x+2}{2x^2+6x+5} dx$ (Delhi 2015C)

32. Find $\int \frac{x+3}{\sqrt{5-4x-2x^2}} dx$. (AI 2015C)

33. Evaluate: $\int \frac{x+2}{\sqrt{x^2+5x+6}} dx$ (AI 2014)

34. Evaluate: $\int \frac{5x-2}{1+2x+3x^2} dx$ (Delhi 2014C) (An)

LA II (5/6 marks)

35. Evaluate: $\int \frac{1}{\cos^4 x + \sin^4 x} dx$ (AI 2014)

36. Evaluate: $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$ (AI 2014) (An)

7.5 Integration by Partial Fractions**SA I (2 marks)**

37. Find $\int \frac{x+1}{(x+2)(x+3)} dx$. (2020)

SA II (3 marks)

38. Find: $\int \frac{x^2}{x^2+6x+12} dx$ (2023)

LA I (4 marks)

39. Find: $\int \frac{x^2}{(x^2+1)(3x^2+4)} dx$ (Term II, 2021-22) (An)

40. Find: $\int \frac{x^3+1}{x^3-x} dx$ (2020) (An)

41. Find: $\int \frac{3x+5}{x^2+3x-18} dx$ (Delhi 2019) (Ap)

42. Evaluate: $\int \frac{x^2+x+1}{(x^2+1)(x+2)} dx$ (2019, AI 2015)

43. Find: $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$ (2018)

44. Find: $\int \frac{2x}{(x^2+1)(x^2+2)^2} dx$. (Delhi 2017) (Ap)

45. Find: $\int \frac{\cos \theta}{(4+\sin^2 \theta)(5-4\cos^2 \theta)} d\theta$ (AI 2017)

46. Find: $\int \frac{x^2}{x^4+x^2-2} dx$ (AI 2016)

47. Find: $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$ (Foreign 2016) (Ap)

48. Find: $\int \frac{dx}{\sin x + \sin 2x}$ (Delhi 2015)

49. Evaluate: $\int \frac{x^2}{(x^2+4)(x^2+9)} dx$ (Foreign 2015) (Ap)

50. Find: $\int \frac{x}{(x-1)^2(x+2)} dx$ (Delhi 2015C)

51. Find: $\int \frac{x}{(x^2+1)(x-1)} dx$ (AI 2015C)

52. Evaluate: $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$ (Delhi 2014C) (Ap)

53. Find: $\int \frac{x^3}{x^4+3x^2+2} dx$ (AI 2014C)

LA II (5/6 marks)

54. Find $\int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$. (Delhi 2014C)

7.6 Integration by Parts**MCQ**

55. $\int e^x \left(\frac{x \log x + 1}{x} \right) dx$ is equal to
 (a) $\log(e^x \log x) + c$ (b) $\frac{e^x}{x} + c$
 (c) $x \log x + e^x + c$ (d) $e^x \log x + c$ (2020) (Ap)

56. $\int \frac{e^x}{x+1} [1+(x+1)\log(x+1)] dx$ equals
 (a) $\frac{e^x}{x+1} + c$ (b) $e^x \frac{x}{x+1} + c$
 (c) $e^x \log(x+1) + e^x + c$
 (d) $e^x \log(x+1) + c$ (2020C)

VSA (1 mark)

57. Find: $\int x^4 \log x dx$ (2020)

SA I (2 marks)

58. Find: $\int \frac{\log x - 3}{(\log x)^4} dx$. (Term II, 2021-22) (An)

59. Find: $\int \sin^{-1}(2x) dx$ (Delhi 2019)

60. Find: $\int x \cdot \tan^{-1} x dx$ (AI 2019) (Ev)

SA II (3 marks)

61. Find: $\int e^x \cdot \sin 2x dx$ (Term II, 2021-22)

LA I (4 marks)

62. Find: $\int \sec^3 x dx$ (2020)

63. Find: $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ (Foreign 2016)

64. Integrate the following w.r.t. x : $\frac{x^2 - 3x + 1}{\sqrt{1-x^2}}$ (Delhi 2015)

65. Find: $\int \frac{\log x}{(x+1)^2} dx$ (AI 2015)

66. Evaluate: $\int e^{2x} \cdot \sin(3x+1) dx$ (Foreign 2015)

67. Find: $\int \frac{(x^2+1)e^x}{(x+1)^2} dx$ (Delhi 2015C) (An)

68. Evaluate: $\int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$ (Foreign 2014)

LA II (5/6 marks)

69. Find: $\int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4} dx$ (AI 2014C)

70. Find: $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx, x \in [0,1]$ (AI 2014C) (An)

7.8 Fundamental Theorem of Calculus**MCQ**

71. $\int_{-1}^1 \frac{|x-2|}{x-2} dx, x \neq 2$ is equal to
(a) 1 (b) -1 (c) 2 (d) -2 (2023)

72. $\int_0^4 (e^{2x} + x) dx$ is equal to
(a) $\frac{15+e^8}{2}$ (b) $\frac{16-e^8}{2}$
(c) $\frac{e^8-15}{2}$ (d) $\frac{-e^8-15}{2}$ (2023)

73. $\int_0^{\pi/8} \tan^2(2x) dx$ is equal to
(a) $\frac{4-\pi}{8}$ (b) $\frac{4+\pi}{8}$
(c) $\frac{4-\pi}{4}$ (d) $\frac{4-\pi}{2}$ (2020) (An)

74. $\int_{-\pi/4}^{\pi/4} \sec^2 x dx$ is equal to

- (a) -1 (b) 0
-
- (c) 1 (d) 2

(2020) (U)

VSA (1 mark)

75. Evaluate: $\int_2^3 3^x dx$ (Delhi 2017)

76. Evaluate: $\int_0^3 \frac{dx}{9+x^2}$ (Delhi 2014) (Ap)

77. Evaluate: $\int_0^{\pi/2} e^x (\sin x - \cos x) dx$ (Delhi 2014)

78. If $f(x) = \int_0^x t \sin t dt$, then write the value of $f'(x)$.
(AI 2014) (Ap)

79. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$, find the value of a . (AI 2014)

80. Evaluate: $\int_0^{\pi/4} \tan x dx$ (Foreign 2014)

81. Evaluate: $\int_0^{\pi/4} \sin 2x dx$ (Foreign 2014)

82. Evaluate: $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ (AI 2014C)

83. Evaluate: $\int_1^2 \frac{x^3-1}{x^2} dx$ (AI 2014C)

SA I (2 marks)

84. Evaluate: $\int_0^1 x^2 e^x dx$ (Term II, 2021-22)

85. Evaluate $\int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2} \right] e^{2x} dx$. (2020) (Ap)

86. Find the value of $\int_0^1 \tan^{-1} \left(\frac{1-2x}{1+x-x^2} \right) dx$. (2020)

LA I (4 marks)

87. Evaluate: $\int_0^{\pi/2} x^2 \sin x dx$ (Delhi 2014C)

7.9 Evaluation of Definite Integrals by Substitution**VSA (1 mark)**

88. Evaluate: $\int_2^4 \frac{x}{x^2+1} dx$ (AI 2014) (An)

89. Evaluate: $\int_e^{e^2} \frac{dx}{x \log x}$. (AI 2014)

90. Evaluate: $\int_0^1 xe^{x^2} dx$ (Foreign 2014)

91. Evaluate: $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$ (AI 2014C)

SA I (2 marks)

92. Find: $\int_{-\frac{\pi}{4}}^0 \frac{1+\tan x}{1-\tan x} dx$ (AI 2019) (Ev)

SA II (3 marks)

93. Evaluate: $\int_{\pi/4}^{\pi/2} e^{2x} \left(\frac{1-\sin 2x}{1-\cos 2x} \right) dx$ (2023)

94. Evaluate: $\int_0^{\pi/2} \sqrt{\sin x} \cos^5 x dx$ (2023)

LA I (4 marks)

95. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx$ (2020C, AI 2014C) (Ev)

96. Evaluate $\int_0^{\pi} e^{2x} \cdot \sin\left(\frac{\pi}{4} + x\right) dx$. (Delhi 2016)

97. Find: $\int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2\sin 2x}}$ (AI 2015)

LA II (5/6 marks)

98. Evaluate: $\int_0^{\pi/4} \frac{\sin x + \cos x}{16+9\sin 2x} dx$ (2018)

99. Evaluate: $\int_0^{\pi/4} \frac{\sin x + \cos x}{9+16\sin 2x} dx$
(Foreign 2014, Delhi 2014C) (Ap)

7.10 Some Properties of Definite Integrals

MCQ

100. In the following question, a statement of Assertion (A) is followed by a statement of Reason (R). Choose the correct answer out of the following choice

Assertion (A): $\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}} dx = 3$

Reason (R): $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

- Both (A) and (R) are true and (R) is the correct explanation of (A).
- Both (A) and (R) are true, but (R) is not the correct explanation of the (A).
- (A) is true and (R) is false.
- (A) is false, but (R) is true. (2023)

VSA (1 mark)

101. Evaluate: $\int_0^{\pi/2} \frac{1}{1+\cot^{5/2} x} dx$ (Term II, 2021-22C)

102. Evaluate: $\int_1^3 |2x-1| dx$ (2020)

103. Evaluate: $\int_{-2}^2 |x| dx$ (2020)

104. Find the value of $\int_1^4 |x-5| dx$. (2020) (An)

SA I (2 marks)

105. Evaluate: $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$ (2021C)

SA II (3 marks)

106. Evaluate: $\int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$ (2023)

107. Evaluate: $\int_{-2}^2 \frac{x^2}{1+5^x} dx$ (2023)

108. Evaluate: $\int_1^4 [|x| + |3-x|] dx$ (Term II, 2021-22)

109. Evaluate: $\int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} dx$ (Term II, 2021-22) (Ap)

LA I (4 marks)

110. Evaluate: $\int_0^{\pi} \frac{x}{9\sin^2 x + 16\cos^2 x} dx$ (Term II, 2021-22)

111. Evaluate: $\int_{-1}^2 |x^3 - x| dx$
(Term II, 2021-22, 2020, Delhi 2016)

112. Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, hence

evaluate $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$. (Delhi 2019)

OR

Evaluate: $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ (Delhi 2017) (Ap)

113. Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ and hence evaluate

$\int_0^1 x^2(1-x)^n dx$. (AI 2019)

114. Evaluate: $\int_0^{3/2} |x \sin \pi x| dx$ (Delhi 2017)

115. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx$ (AI 2017, Foreign 2014, Delhi 2014C)

116. Evaluate: $\int_1^4 (|x-1| + |x-2| + |x-4|) dx$ (AI 2017)

117. Evaluate: $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ (AI 2016)

OR

Show that: $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$ (AI 2014C) (Ap)

118. Evaluate: $\int_0^{3/2} |x \cos \pi x| dx$ (AI 2016)

119. Evaluate: $\int_0^{\pi} \frac{x}{1 + \sin \alpha \sin x} dx$ (Foreign 2016)

120. Evaluate: $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$ (Delhi 2015) (Ev)

121. Evaluate: $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$ (Foreign 2015)

122. Evaluate: $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ (Delhi 2015C) (Ap)

123. Evaluate: $\int_0^{\pi/4} \log(1+\tan x) dx$ (AI 2015C)

124. Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx$ (Delhi 2014)

125. Evaluate: $\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x} dx$ (AI 2014)

LA II (5/6 marks)

126. Evaluate: $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ (Delhi 2014)

127. Evaluate: $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}$ (Delhi 2014)

128. Evaluate: $\int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ (Foreign 2014) (An)

CBSE Sample Questions

7.3 Methods of Integration

SA I (2 marks)

1. Find $\int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$. (2020-21)

7.4 Integrals of Some Particular Functions

SA I (2 marks)

2. Find $\int \frac{\sin 2x}{\sqrt{9 - \cos^4 x}} dx$. (2020-21)

7.5 Integration by Partial Fractions

SA I (2 marks)

3. Find: $\int \frac{x+1}{(x^2+1)x} dx$. (2021-22) (An)

SA II (3 marks)

4. Find $\int \frac{(x^3+x+1)}{(x^2-1)} dx$ (2022-23) (Ev)

5. Evaluate: $\int_0^4 |x-1| dx$ (2022-23) (Ev)

6. Find $\int \frac{x^2+1}{(x^2+2)(x^2+3)} dx$. (2020-21) (Ap)

7.6 Integration by Parts

VSA (1 mark)

7. Find $\int e^x (1 - \cot x + \operatorname{cosec}^2 x) dx$. (2020-21)

SA I (2 marks)

8. Find $\int \frac{\log x}{(1+\log x)^2} dx$. (2021-22) (Ev)

7.10 Some Properties of Definite Integrals

VSA (1 mark)

9. Evaluate: $\int_{-\pi/2}^{\pi/2} x^2 \sin x dx$ (2020-21) (Ev)

SA I (2 marks)

10. Evaluate: $\int_0^1 x(1-x)^n dx$ (2020-21)

LA I (4 marks)

11. Evaluate: $\int_{-1}^2 |x^3 - 3x^2 + 2x| dx$ (2021-22) (An)

Previous Years' CBSE Board Questions

1. (b): Let $I = \int \frac{\sec x}{\sec x - \tan x} dx$

$$= \int \frac{\sec x(\sec x + \tan x)}{(\sec x - \tan x)(\sec x + \tan x)} dx = \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec^2 x - \tan^2 x} \right) dx$$

$$= \int \sec^2 x dx + \int \sec x \tan x dx \quad [\because \sec^2 x - \tan^2 x = 1]$$

$$= \tan x + \sec x + C$$

2. We have, $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

$$= \int \frac{\sin^2 x}{\sin x \cos x} dx - \int \frac{\cos^2 x}{\sin x \cos x} dx$$

$$= \int \tan x dx - \int \cot x dx$$

$$= \ln|\sec x| - \ln|\sin x| + C = \ln \left| \frac{1}{\sin x \cos x} \right| + C$$

$$= \ln \left| \frac{2}{2 \sin x \cos x} \right| + C = \ln|2 \operatorname{cosec} 2x| + C$$

3. The antiderivative of $3\sqrt{x} + \frac{1}{\sqrt{x}}$

$$= \int \left(3\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx = 3 \int x^{1/2} dx + \int x^{-1/2} dx$$

$$= 3 \cdot \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = 2x\sqrt{x} + 2\sqrt{x} + C = 2\sqrt{x}(x+1) + C$$

4. We have, $\int \cos^{-1}(\sin x) dx = \int \cos^{-1} \left[\cos \left(\frac{\pi}{2} - x \right) \right] dx$

$$= \int \left(\frac{\pi}{2} - x \right) dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$$

5. We have, $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x} dx$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x) dx = \tan x - \cot x + C$$

6. We have, $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} dx$

$$= \int \sec^2 x dx - \int \operatorname{cosec}^2 x dx$$

$$= \tan x + \cot x + C$$

7. Let $I = \int (\sqrt{1 - \sin 2x}) dx$

$$= \int \sqrt{\cos^2 x + \sin^2 x - 2 \sin x \cos x} dx = \int (\cos x - \sin x) dx$$

Since, $\frac{\pi}{4} < x < \frac{\pi}{2}$, so we get

$$I = \int (\sin x - \cos x) dx$$

$$= -(\cos x + \sin x) + C$$

8. Let $I = \int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} dx$

$$= \int \frac{\cos^2 x - \sin^2 x + 2 \sin^2 x}{\cos^2 x} dx$$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx$$

$$= \tan x + C$$

9. (b): Let $I = \int e^{5 \log x} dx$

$$= \int e^{\log x^5} dx = \int x^5 dx \quad [\because e^{\log x} = x]$$

$$= \frac{x^6}{6} + C$$

10. (a): Let $I = \int x^2 e^{x^3} dx$

Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore I = \int e^t \frac{dt}{3} = \frac{1}{3} e^t + C = \frac{1}{3} e^{x^3} + C$$

11. (a): Let $I = \int \frac{e^x(1+x)}{\cos^2(xe^x)} dx$

Put $xe^x = t$

$$\Rightarrow (xe^x + e^x) dx = dt$$

$$\Rightarrow e^x(x+1) dx = dt$$

$$\therefore I = \int \frac{dt}{\cos^2 t} = \int \sec^2 t dt = \tan t + c = \tan(xe^x) + c$$

12. Let $I = \int \frac{2x}{\sqrt[3]{x^2+1}} dx$

Put $x^2 + 1 = z \Rightarrow 2x dx = dz$

$$\therefore I = \int \frac{dz}{(z)^{1/3}} = \int z^{-1/3} dz = \frac{z^{(-1/3)+1}}{(-1/3)+1} + C$$

$$= \frac{3}{2} (x^2 + 1)^{2/3} + C$$

13. Let $I = \int \frac{\sin^6 x}{\cos^8 x} dx = \int \frac{\sin^6 x}{\cos^6 x \cdot \cos^2 x} dx$

$$= \int \tan^6 x \sec^2 x dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I = \int t^6 dt = \frac{t^7}{7} + C = \frac{1}{7} \tan^7 x + C$$

14. $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$...(i)

Put $x - 2 = 2 \sin \theta$

$$\Rightarrow dx = 2 \cos \theta d\theta$$

$$= \int \frac{2 \cos \theta}{\sqrt{4 - (2 \sin \theta)^2}} d\theta$$

Now, $4 - (2 \sin \theta)^2 = 4 - 4 \sin^2 \theta = 4(1 - \sin^2 \theta) = 4 \cos^2 \theta$

$$= \int \frac{2 \cos \theta d\theta}{\sqrt{4 \cos^2 \theta}} = \int \frac{2 \cos \theta d\theta}{2 \cos \theta} = \theta + c$$

$$= \sin^{-1} \left(\frac{x-2}{2} \right) + c, \quad \left[\text{From (i), } \theta = \sin^{-1} \left(\frac{x-2}{2} \right) \right]$$

where c is an arbitrary constant.

15. Let $I = \int \frac{\sin x}{\sin(x-2a)} dx$

Put $x - 2a = t$

$$\Rightarrow x = 2a + t \Rightarrow dx = dt$$

$$\therefore I = \int \frac{\sin(t+2a)}{\sin t} dx$$

$$= \int \frac{(\sin t \cos 2a + \cos t \sin 2a)}{\sin t} dx = \int (\cos 2a + \cot t \cdot \sin 2a) dx$$

$$= t \cos 2a + \sin 2a \log |\sin t| + c$$

$$= (x - 2a) \cos 2a + \sin 2a \log |\sin(x - 2a)| + c$$

16. Let $I = \int \frac{1}{e^x + 1} dx$

Put $e^x + 1 = t \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{e^x} = \frac{dt}{t-1}$

$$\therefore I = \int \frac{dt}{t(t-1)} = \int \frac{t-(t-1)}{t(t-1)} dt$$

$$= \int \frac{1}{t-1} dt - \int \frac{1}{t} dt = \log(t-1) - \log t + C$$

$$= \log e^x - \log(e^x + 1) + c = \log \frac{e^x}{e^x + 1} + c$$

17. $\int \frac{2x}{(x^2+1)(x^2+2)} dx$

Let $x^2 + 2 = t \Rightarrow 2x dx = dt$

$$\Rightarrow \int \frac{2x}{(x^2+1)(x^2+2)} dx = \int \frac{dt}{(t-1)t} = \int \frac{(1-t+t) dt}{(t-1)t}$$

$$= -\int \frac{(t-1) dt}{(t-1)t} + \int \frac{t}{(t-1)t} dt = -\int \frac{1}{t} dt + \int \frac{1}{(t-1)} dt$$

$$= -\log |t| + \log |t-1| + \log C$$

$$= \log \left| \frac{t-1}{t} \right| + \log C = \log \left| \frac{x^2+1}{x^2+2} \right| + C$$

where C is an arbitrary constant.

18. We have, $\int \frac{\cos(x+a)}{\sin(x+b)} dx = \int \frac{\cos(x+b+a-b)}{\sin(x+b)} dx$

$$= \int \frac{\cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b)}{\sin(x+b)} dx$$

$$= \cos(a-b) \int \frac{\cos(x+b)}{\sin(x+b)} dx - \sin(a-b) \int dx$$

$$= \cos(a-b) \log |\sin(x+b)| - x \sin(a-b) + C$$

Commonly Made Mistake ⚠

Using formula for $\cos(A+B) = \cos A \cos B - \sin A \sin B$ and $\cos(A-B) = \cos A \cos B + \sin A \sin B$

19. Let $I = \int \frac{(3 \sin \theta - 2) \cos \theta}{5 - \cos^2 \theta - 4 \sin \theta} d\theta$

(Put $\cos^2 \theta = 1 - \sin^2 \theta$)

$$= 3 \int \frac{\sin \theta \cos \theta}{4 + \sin^2 \theta - 4 \sin \theta} d\theta - 2 \int \frac{\cos \theta}{4 + \sin^2 \theta - 4 \sin \theta} d\theta$$

$$= 3I_1 - 2I_2 \text{ (say)}$$

Now, $I_1 = \int \frac{\sin \theta \cos \theta}{4 + \sin^2 \theta - 4 \sin \theta} d\theta$

Put $\sin^2 \theta = t \Rightarrow 2 \sin \theta \cos \theta d\theta = dt$

$$\therefore I_1 = \frac{1}{2} \int \frac{dt}{4+t-4\sqrt{t}} = \frac{1}{2} \int \frac{dt}{(\sqrt{t}-2)^2}$$

Put $\sqrt{t}-2=u \Rightarrow \sqrt{t}=u+2$

$$\Rightarrow \frac{1}{2\sqrt{t}} dt = du \Rightarrow dt = 2(u+2)du$$

$$\therefore I_1 = \int \frac{(u+2)}{u^2} du = \int \frac{du}{u} + 2 \int \frac{du}{u^2}$$

$$= \log u - \frac{2}{u} + C_1 = \log(\sqrt{t}-2) - \frac{2}{\sqrt{t}-2} + C_1$$

$$= \log(\sin \theta - 2) - \frac{2}{\sin \theta - 2} + C_1$$

...(ii)

Also, $I_2 = \int \frac{\cos \theta}{4 + \sin^2 \theta - 4 \sin \theta} d\theta$

Put $\sin \theta = m \Rightarrow \cos \theta d\theta = dm$

$$\therefore I_2 = \int \frac{dm}{4+m^2-4m} = \int \frac{dm}{(m-2)^2}$$

$$= \frac{-1}{m-2} + C_2 = \frac{-1}{\sin \theta - 2} + C_2$$

...(iii)

From (i), (ii) and (iii), we get

$$I = 3 \log(\sin \theta - 2) - \frac{6}{\sin \theta - 2} + \frac{2}{\sin \theta - 2} + C,$$

where $C = 3C_1 - 2C_2$

$$\Rightarrow I = 3 \log(\sin \theta - 2) - \frac{4}{\sin \theta - 2} + C$$

20. Let $I = \int \frac{\sin(x-a)}{\sin(x+a)} dx = \int \frac{\sin(x+a-2a)}{\sin(x+a)} dx$

$$= \int \left(\frac{\sin(x+a)\cos 2a - \cos(x+a)\sin 2a}{\sin(x+a)} \right) dx$$

$$\Rightarrow I = \cos 2a \int dx - \sin 2a \int \frac{\cos(x+a)}{\sin(x+a)} dx$$

Put $\sin(x+a) = t \Rightarrow \cos(x+a) dx = dt$

$$\Rightarrow I = \cos 2a \int dx - \sin 2a \int \frac{dt}{t}$$

$$\Rightarrow I = x \cos 2a - \sin 2a \log |t| + C$$

$$\Rightarrow I = x \cos 2a - \sin 2a \log |\sin(x+a)| + C$$

21. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$

$$= \int \frac{\sin^6 x}{\sin^2 x \cdot \cos^2 x} dx + \int \frac{\cos^6 x}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \frac{\sin^4 x}{\cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x} dx$$

$$= \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x} dx + \int \frac{\cos^2 x(1-\sin^2 x)}{\sin^2 x} dx$$

$$= \int \tan^2 x dx - \int \sin^2 x dx + \int \cot^2 x dx - \int \cos^2 x dx$$

$$= \int (\sec^2 x - 1) dx - \int \sin^2 x dx + \int (\operatorname{cosec}^2 x - 1) dx - \int (1 - \sin^2 x) dx$$

$$= \tan x - x + (-\cot x) - x + C = \tan x - \cot x - 3x + C$$

22. Let $I = \int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \int \frac{dx}{\sqrt{\frac{9}{4}-x^2}}$

$$= \frac{1}{2} \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}} = \frac{1}{2} \sin^{-1} \left(\frac{x}{\frac{3}{2}} \right) + C = \frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C$$

23. Let $I = \int \frac{dx}{9+4x^2} = \frac{1}{4} \int \frac{dx}{x^2 + \frac{9}{4}} = \frac{1}{4} \int \frac{dx}{x^2 + \left(\frac{3}{2}\right)^2}$

...(i)

$$= \frac{1}{4} \cdot \frac{2}{3} \tan^{-1}\left(\frac{2x}{3}\right) + C = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + C$$

24. Let $I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}} dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{t^2 + 4}} = \log|t + \sqrt{t^2 + 4}| + C$$

$$= \log|\tan x + \sqrt{\tan^2 x + 4}| + C$$

25. Let $I = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{7}{2}-1-2x-x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{7}}{2}\right)^2 - (x+1)^2}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x+1}{\frac{\sqrt{7}}{2}}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}\left[\frac{\sqrt{2}}{\sqrt{7}}(x+1)\right] + C$$

26. We have, $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{x^2+4x+4+4}$

$$= \int \frac{dx}{(x+2)^2 + (2)^2} = \frac{1}{2} \tan^{-1}\left(\frac{x+2}{2}\right) + C$$

27. Let $I = \int \frac{dx}{5-8x-x^2}$

$$= \int \frac{dx}{5+16-16-8x-x^2} = \int \frac{dx}{21-(x+4)^2}$$

$$= \int \frac{dx}{(\sqrt{21})^2 - (x+4)^2} = \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21}+x+4}{\sqrt{21}-x-4}\right| + C$$

$$\left[\because \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C \right]$$

28. Let $I = \int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$

Putting $e^x = t \Rightarrow e^x dx = dt$, we get

$$I = \int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{9-(t^2+4t+4)}} = \int \frac{dt}{\sqrt{3^2-(t+2)^2}}$$

$$= \sin^{-1}\left(\frac{t+2}{3}\right) + C = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C$$

29. Let $I = \int \frac{\sqrt{x}}{\sqrt{a^3-x^3}} dx$

Put $x^{3/2} = t \Rightarrow \frac{3}{2} x^{1/2} dx = dt$

$$\therefore I = \frac{2}{3} \int \frac{dt}{\sqrt{a^3-t^2}} = \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2-t^2}}$$

$$= \frac{2}{3} \left[\sin^{-1}\left(\frac{t}{a^{3/2}}\right) \right] + C = \frac{2}{3} \left[\sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) \right] + C$$

$$= \frac{2}{3} \sin^{-1}\left(\frac{x}{a}\right)^{3/2} + C$$

30. Let $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx$

$$= \int \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}} \right) dx = \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx = \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{\sin 2x + 1 - 1}} dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (1 - \sin 2x)}} dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin^2 x + \cos^2 x - 2 \sin x \cos x)}} dx$$

$$= \sqrt{2} \int \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$

$$\therefore I = \sqrt{2} \int \frac{dt}{\sqrt{1-t^2}} = \sqrt{2} \sin^{-1} t + C$$

$$= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C$$

31. Let $I = \int \frac{x+2}{2x^2+6x+5} dx$

Let $x+2 = A \left[\frac{d}{dx}(2x^2+6x+5) \right] + B$

$$\Rightarrow x+2 = A(4x+6) + B$$

Equating coefficients of x and constant terms, we get

$$4A = 1 \Rightarrow A = 1/4 \text{ and } 6A + B = 2 \Rightarrow B = 1/2$$

$$\therefore I = \frac{1}{4} \int \frac{4x+6}{2x^2+6x+5} dx + \frac{1}{2} \int \frac{dx}{2x^2+6x+5}$$

$$= \frac{1}{4} \log|2x^2+6x+5| + \frac{1}{4} \int \frac{dx}{x^2+3x+\frac{5}{2}}$$

$$= \frac{1}{4} \log|2x^2+6x+5| + \frac{1}{4} \int \frac{dx}{x^2+3x+\frac{9}{4}-\frac{9}{4}+\frac{5}{2}}$$

$$= \frac{1}{4} \log|2x^2+6x+5| + \frac{1}{4} \int \frac{dx}{\left(x+\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4} \log|2x^2+6x+5| + \frac{1}{2} \tan^{-1}\left(\frac{x+\frac{3}{2}}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{4} \log|2x^2+6x+5| + \frac{1}{2} \tan^{-1}(2x+3) + C$$

32. Let $I = \int \frac{x+3}{\sqrt{5-4x-2x^2}} dx$

Let $x+3 = A \frac{d}{dx}(5-4x-2x^2) + B = A(-4-4x) + B$

On comparing the coefficients of like term, we get

$$-4A = 1 \Rightarrow A = -\frac{1}{4} \text{ and } -4A + B = 3 \Rightarrow B = 2$$

$$x+3 = -\frac{1}{4}(-4-4x) + 2$$

$$\Rightarrow I = \int \frac{-\frac{1}{4}(-4-4x) + 2}{\sqrt{5-4x-2x^2}} dx$$

$$= \frac{-1}{4} \int \frac{-4-4x}{\sqrt{5-4x-2x^2}} dx + 2 \int \frac{1}{\sqrt{5-4x-2x^2}} dx$$

$$= -\frac{1}{4} I_1 + 2I_2$$

$$\text{where } I_1 = \int \frac{-4-4x}{\sqrt{5-4x-2x^2}} dx$$

$$\text{Put } 5-4x-2x^2 = t \Rightarrow (-4-4x)dx = dt$$

$$\therefore I_1 = \int \frac{dt}{t} = \int t^{-1/2} dt = 2\sqrt{t} + C_1$$

$$= 2\sqrt{5-4x-2x^2} + C_1$$

$$\text{and } I_2 = \int \frac{dx}{\sqrt{5-4x-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\frac{5}{2}-2x-x^2}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 - (x+1)^2}} = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{x+1}{\sqrt{7/2}} \right) + C_2$$

From (i), (ii) and (iii), we get

$$I = -\frac{1}{4} \cdot 2\sqrt{5-4x-2x^2} + 2 \cdot \frac{1}{\sqrt{2}} \sin^{-1} \left[\frac{\sqrt{2}}{\sqrt{7}}(x+1) \right] + C$$

where $C = C_1 + C_2$

$$\Rightarrow I = -\frac{1}{2} \sqrt{5-4x-2x^2} + \sqrt{2} \sin^{-1} \left[\frac{\sqrt{2}}{\sqrt{7}}(x+1) \right] + C$$

$$33. \text{ Let } I = \int \frac{x+2}{\sqrt{x^2+5x+6}} dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}} dx$$

$$= \frac{1}{2} \int (x^2+5x+6)^{-1/2} (2x+5) dx - \frac{1}{2} \int \frac{dx}{\sqrt{x^2+5x+6}}$$

$$\text{Put } x^2+5x+6 = t \Rightarrow (2x+5) dx = dt$$

$$\Rightarrow I = \frac{1}{2} \int t^{-1/2} dt - \frac{1}{2} \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} + C$$

$$= \frac{1}{2} \frac{t^{1/2}}{\frac{1}{2}} - \frac{1}{2} \log \left| \left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + C$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2} \log \left| x + \frac{5}{2} + \sqrt{x^2+5x+6} \right| + C$$

Concept Applied

$$\Rightarrow \ln \int \frac{px+q}{ax^2+bx+c}$$

$$\text{substitute } px+q = A \frac{d}{dx}(ax^2+bx+c) + B.$$

$$34. \text{ Let } I = \int \frac{5x-2}{1+2x+3x^2} dx$$

$$\text{Let } 5x-2 = A \frac{d}{dx}(1+2x+3x^2) + B = A(2+6x) + B$$

On comparing the coefficients of like terms, we get

$$6A=5 \Rightarrow A = \frac{5}{6} \text{ and } 2A+B=-2 \Rightarrow B = \frac{-11}{3}$$

$$\therefore I = \int \frac{\frac{5}{6} d(1+2x+3x^2) - \frac{11}{3}}{1+2x+3x^2} dx$$

$$= \frac{5}{6} \int \frac{d(1+2x+3x^2)}{1+2x+3x^2} dx - \frac{11}{3} \int \frac{dx}{1+2x+3x^2}$$

$$\dots(i) = \frac{5}{6} \log|1+2x+3x^2| - \frac{11}{3 \cdot 3} \int \frac{dx}{x^2 + \frac{2}{3}x + \frac{1}{3}}$$

$$= \frac{5}{6} \log|1+2x+3x^2| - \frac{11}{9} \int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$\dots(ii) = \frac{5}{6} \log|1+2x+3x^2| - \frac{11}{9} \cdot \frac{1}{\sqrt{2/3}} \tan^{-1} \left(\frac{x+\frac{1}{3}}{\sqrt{2/3}} \right) + C$$

$$\dots(iii) = \frac{5}{6} \log|1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C$$

$$35. \text{ Let } I = \int \frac{1}{\cos^4 x + \sin^4 x} dx = \int \frac{\sec^4 x}{1 + \tan^4 x} dx$$

$$= \int \frac{(\tan^2 x + 1) \sec^2 x}{1 + \tan^4 x} dx$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{t^2+1}{t^4+1} dt = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2 + 2} dt$$

$$\text{Put } t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dy$$

$$\therefore I = \int \frac{dy}{y^2 + (\sqrt{2})^2} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{y}{\sqrt{2}} \right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x - \cot x}{\sqrt{2}} \right) + C$$

$$36. \text{ Let } I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

$$= \int \frac{\sec^4 x dx}{\tan^4 x + \tan^2 x + 1} = \int \frac{(1 + \tan^2 x) \sec^2 x dx}{\tan^4 x + \tan^2 x + 1}$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{1+t^2}{t^4+t^2+1} dt = \int \frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}} dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2 + 3} dt$$

$$\text{Put } t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dy$$

$$\text{Thus, } I = \int \frac{dy}{y^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{y}{\sqrt{3}} \right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t-\frac{1}{t}}{\sqrt{3}} \right) + C = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\tan x - \cot x}{\sqrt{3}} \right) + C$$

$$37. \text{ Let } I = \int \frac{(x+1)}{(x+2)(x+3)} dx$$

$$\text{Also let, } \frac{(x+1)}{(x+2)(x+3)} = \frac{A}{(x+2)} + \frac{B}{(x+3)}$$

$$\Rightarrow x+1 = A(x+3) + B(x+2)$$

...(i)

Putting $x = -3$ in (i), we get

$$-B = -3 + 1 = -2 \Rightarrow B = 2$$

Putting $x = -2$ in (i), we get

$$A = -2 + 1 = -1$$

$$\therefore I = \int \frac{-1}{(x+2)} dx + 2 \int \frac{1}{(x+3)} dx$$

$$= -\log(x+2) + 2\log(x+3) + C$$

Answer Tips

⇒ Form of rational fraction to the partial fraction.

$$\frac{px+q}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

38. Let $I = \int \frac{x^2}{x^2+6x+12} dx$

$$= \int \frac{x^2+6x+12-(6x+12)}{x^2+6x+12} dx = \int dx - \int \frac{6x+12}{x^2+6x+12} dx$$

$$= x - 3 \int \frac{2x}{x^2+6x+12} dx - 12 \int \frac{dx}{x^2+6x+12}$$

$$= x - 3 \int \frac{2x+6-6}{x^2+6x+12} dx - 12 \int \frac{dx}{x^2+6x+12}$$

$$= x - 3 \int \frac{2x+6}{x^2+6x+12} dx - 18 \int \frac{dx}{x^2+6x+12}$$

$$= x - 3I_1 - 18I_2$$

Consider, $I_1 = \int \frac{2x+6}{x^2+6x+12} dx$

Put $x^2+6x+12 = t \Rightarrow (2x+6)dx = dt$

$$\therefore I_1 = \int \frac{dt}{t} = \log|t| + C = \log|x^2+6x+12| + C_1$$

and $I_2 = \int \frac{dx}{x^2+6x+12}$

$$= \int \frac{dx}{(x+3)^2+(\sqrt{3})^2} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C_2$$

$$\therefore I = x - 3\log|x^2+6x+12| + 6\sqrt{3} \tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C$$

39. Let $I = \int \frac{x^2}{(x^2+1)(3x^2+4)} dx$

Let $x^2 = y$

$$\text{So, } \frac{x^2}{(x^2+1)(3x^2+4)} = \frac{y}{(y+1)(3y+4)}$$

$$= \frac{A}{y+1} + \frac{B}{3y+4} = \frac{A(3y+4)+B(y+1)}{(y+1)(3y+4)}$$

$$\Rightarrow y = A(3y+4) + B(y+1)$$

$$\Rightarrow y = (3A+B)y + (4A+B)$$

Equating the like coefficients, we get

$$3A+B = 1 \text{ and } 4A+B = 0$$

On solving we get $A = -1, B = 4$

$$\text{So, } I = \int \left[\frac{-1}{x^2+1} + \frac{4}{3x^2+4} \right] dx$$

$$= -\int \frac{1}{1+x^2} dx + \frac{4}{3} \int \frac{1}{x^2+\frac{4}{3}} dx$$

$$= -\tan^{-1}x + \frac{4}{3} \times \frac{\sqrt{3}}{2} \tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$$

$$= -\tan^{-1}x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C$$

40. Let $I = \int \frac{x^3+1}{x^3-x} dx = \int \frac{(x+1)(x^2-x+1)}{x(x^2-1)} dx$

$$= \int \frac{(x^2-x+1)}{x(x-1)} dx = \int \left[1 + \frac{1}{x(x-1)} \right] dx \quad \dots(i)$$

Consider, $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$

$$\Rightarrow \frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)}$$

$$\Rightarrow 1 = A(x-1) + Bx$$

On solving, we get $A = -1, B = 1$

$$\Rightarrow \frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$$

$$\therefore I = \int \left(1 + \frac{1}{x-1} - \frac{1}{x} \right) dx = x + \log|x-1| - \log|x| + C$$

41. Let $I = \int \frac{3x+5}{x^2+3x-18} dx = \int \frac{3x+5}{(x+6)(x-3)} dx$

Let $\frac{3x+5}{(x+6)(x-3)} = \frac{A}{x+6} + \frac{B}{x-3}$

$$\Rightarrow 3x+5 = A(x-3) + B(x+6)$$

...

Putting $x = 3$ in (i), we get $9B = 14 \Rightarrow B = \frac{14}{9}$

Putting $x = -6$ in (i), we get $-9A = -13 \Rightarrow A = \frac{13}{9}$

$$\therefore I = \frac{13}{9} \int \frac{1}{(x+6)} dx + \frac{14}{9} \int \frac{1}{(x-3)} dx$$

$$= \frac{13}{9} \log|x+6| + \frac{14}{9} \log|x-3| + C$$

42. Let $I = \int \frac{x^2+x+1}{(x^2+1)(x+2)} dx \quad \dots(i)$

Let $\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+2} \quad \dots(ii)$

$$\Rightarrow x^2+x+1 = (Ax+B)(x+2) + C(x^2+1) \quad \dots(iii)$$

Put $x = 0, 1$ and -2 in (iii), We get

$$1 = 2B + C; 3 = 3(A+B) + 2C \text{ and } 3 = 5C$$

$$\Rightarrow C = \frac{3}{5}, B = \frac{1}{5} \text{ and } A = \frac{2}{5}$$

From (ii), we get

$$\frac{x^2+x+1}{(x^2+1)(x+2)} = \frac{\left(\frac{2}{5}x+\frac{1}{5}\right)}{x^2+1} + \frac{\frac{3}{5}}{x+2}$$

$$= \frac{1}{5} \frac{2x+1}{x^2+1} + \frac{3}{5} \frac{1}{x+2}$$

$$\therefore I = \frac{1}{5} \int \frac{2x+1}{x^2+1} dx + \frac{3}{5} \int \frac{1}{x+2} dx$$

$$= \frac{1}{5} \left[\int \frac{2x}{x^2+1} dx + \int \frac{dx}{x^2+1} \right] + \frac{3}{5} \int \frac{dx}{x+2}$$

$$= \frac{1}{5} [\log|x^2+1| + \tan^{-1} x] + \frac{3}{5} \log|x+2| + C_1$$

43. Let $I = \int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)} dx$

Put $\sin x = t \Rightarrow \cos x dx = dt$

$$\therefore I = 2 \int \frac{1}{(1-t)(1+t^2)} dt$$

...(i)

Now, $\frac{1}{(1-t)(1+t^2)} = \frac{A}{1-t} + \frac{Bt+C}{1+t^2}$

$$\Rightarrow 1 = A(1+t^2) + (Bt+C)(1-t)$$

$$\Rightarrow 1 = A + At^2 + Bt - Bt^2 + C - Ct$$

$$\Rightarrow 1 = (A+C) + t^2(A-B) + t(B-C)$$

On equating coefficients of like terms on both sides, we get

$$A+C=1, A-B=0 \Rightarrow A=B$$

$$B-C=0 \Rightarrow B=C$$

$$\therefore A=B=C$$

$$\therefore A = \frac{1}{2}, B = \frac{1}{2}, C = \frac{1}{2}$$

$$\therefore \frac{1}{(1-t)(1+t^2)} = \frac{1}{2(1-t)} + \frac{1}{2} \frac{t}{1+t^2} + \frac{1}{2(1+t^2)}$$

Put above equation in (i), we get

$$I = 2 \left[\int \frac{1}{2(1-t)} dt + \int \frac{1}{2} \frac{t}{1+t^2} dt + \int \frac{1}{2(1+t^2)} dt \right]$$

$$= \int \frac{dt}{1-t} + \int \frac{t}{1+t^2} dt + \int \frac{dt}{1+t^2}$$

$$= -\log(1-t) + \frac{1}{2} \log(1+t^2) + \tan^{-1}(t) + C$$

$$= -\log(1-\sin x) + \frac{1}{2} \log(1+\sin^2 x) + \tan^{-1}(\sin x) + C$$

$$= -\log(1-\sin x) + \log(1+\sin^2 x)^{1/2} + \tan^{-1}(\sin x) + C$$

$$= \log \frac{\sqrt{1+\sin^2 x}}{1-\sin x} + \tan^{-1}(\sin x) + C$$

44. Let $I = \int \frac{2x}{(x^2+1)(x^2+2)^2} dx$

Put $x^2 = y \Rightarrow 2x dx = dy$

$$\therefore I = \int \frac{dy}{(y+1)(y+2)^2}$$

Let $\frac{1}{(y+1)(y+2)^2} = \frac{A}{y+1} + \frac{B}{y+2} + \frac{C}{(y+2)^2}$

$$\Rightarrow 1 = A(y+2)^2 + B(y+1)(y+2) + C(y+1)$$

Putting $y = -1$, in (i), we get $1 = A$

Putting $y = -2$, in (i), we get $1 = -C \Rightarrow C = -1$

Putting $y = 0$, in (i), we get $1 = 4A + 2B + C$

$$\Rightarrow B = \frac{1-4+1}{2} = -1$$

$$\therefore I = \int \left[\frac{1}{y+1} - \frac{1}{y+2} - \frac{1}{(y+2)^2} \right] dy$$

$$= \log(y+1) - \log(y+2) + \frac{1}{y+2} + C$$

$$= \log \left(\frac{y+1}{y+2} \right) + \frac{1}{y+2} + C$$

$$= \log \left(\frac{x^2+1}{x^2+2} \right) + \frac{1}{x^2+2} + C$$

[$\because y = x^2$]

45. $I = \int \frac{\cos \theta}{(4+\sin^2 \theta)(5-4\cos^2 \theta)} d\theta$

$$= \int \frac{\cos \theta}{(4+\sin^2 \theta)(1+4\sin^2 \theta)} d\theta$$

Let $\sin \theta = t \Rightarrow \cos \theta d\theta = dt$

$$\therefore I = \int \frac{1}{(4+t^2)(1+4t^2)} dt$$

Consider, $\frac{1}{(4+t^2)(1+4t^2)} = \frac{At+B}{4+t^2} + \frac{Ct+D}{1+4t^2}$

(Using partial fractions)

$$1 = (At+B)(1+4t^2) + (Ct+D)(4+t^2)$$

$$= At + B + 4At^3 + 4Bt^2 + 4Ct + Ct^3 + 4D + Dt^2$$

$$= (4A+C)t^3 + (4B+D)t^2 + (A+4C)t + (B+4D)$$

On comparing the coefficients of like terms, we get

$$4A + C = 0$$

...(i)

$$4B + D = 0$$

...(ii)

$$A + 4C = 0$$

...(iii)

$$B + 4D = 1$$

...(iv)

Solving (i) & (iii), we get

$$A = 0 \text{ and } C = 0$$

Solving (ii) & (iv), we get

$$B = \frac{-1}{15} \text{ and } D = \frac{4}{15}$$

$$\therefore \frac{1}{(4+t^2)(1+4t^2)} = \frac{-1/15}{4+t^2} + \frac{4/15}{1+4t^2}$$

$$\therefore I = -\frac{1}{15} \int \frac{1}{4+t^2} dt + \frac{4}{15} \times \frac{1}{4} \int \frac{1}{1+t^2} dt$$

$$= -\frac{1}{15} \times \frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right) + \frac{1}{15} \times \frac{1}{1/2} \tan^{-1} \left(\frac{t}{1/2} \right) + C$$

$$= -\frac{1}{30} \tan^{-1} \left(\frac{t}{2} \right) + \frac{2}{15} \tan^{-1}(2t) + C$$

$$= \frac{2}{15} \tan^{-1}(2\sin \theta) - \frac{1}{30} \tan^{-1} \left(\frac{\sin \theta}{2} \right) + C$$

46. Let $I = \int \frac{x^2}{x^4+x^2-2} dx = \int \frac{x^2}{(x^2-1)(x^2+2)} dx$

Let $x^2 = z$

$$\therefore \frac{x^2}{(x^2-1)(x^2+2)} = \frac{z}{(z-1)(z+2)}$$

...(i)

Using partial fractions, we have

$$\frac{z}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$\Rightarrow z = A(z+2) + B(z-1)$$

when $z = 1$, we get $A = \frac{1}{3}$

and when $z = -2$, we get $B = \frac{2}{3}$

$$\therefore I = \int \frac{x^2}{(x^2-1)(x^2+2)} dx$$

$$\begin{aligned}
 &= \int \frac{1/3}{(x^2-1)} dx + \int \frac{2/3}{(x^2+2)} dx \\
 &= \frac{1}{3} \int \frac{1}{x^2-1} dx + \frac{2}{3} \int \frac{1}{x^2+(\sqrt{2})^2} dx \\
 &= \frac{1}{3} \times \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| + \frac{2}{3} \times \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + C \\
 &= \frac{1}{6} \log \left| \frac{x-1}{x+1} \right| + \frac{\sqrt{2}}{3} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + C
 \end{aligned}$$

Concept Applied

$$\Rightarrow \int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

47. Let $I = \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx$

Let $x^2 = t$

$$\begin{aligned}
 \therefore \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} &= \frac{(t+1)(t+4)}{(t+3)(t-5)} \\
 &= \frac{t^2+5t+4}{(t+3)(t-5)} = 1 + \frac{7t+19}{(t+3)(t-5)}
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } \frac{7t+19}{(t+3)(t-5)} &= \frac{A}{t+3} + \frac{B}{t-5} \\
 \Rightarrow 7t+19 &= A(t-5) + B(t+3)
 \end{aligned}$$

Putting $t = 5$, we get $B = \frac{27}{4}$

Putting $t = -3$, we get $A = \frac{1}{4}$

$$\therefore \frac{t^2+5t+4}{(t+3)(t-5)} = 1 + \frac{1}{4(t+3)} + \frac{27}{4(t-5)}$$

$$\begin{aligned}
 \Rightarrow I &= \int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)} dx = \int dx + \frac{1}{4} \int \frac{1}{(x^2+3)} dx \\
 &\quad + \frac{27}{4} \int \frac{1}{(x^2-5)} dx
 \end{aligned}$$

$$= x + \frac{1}{4\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{27}{4} \times \frac{1}{2\sqrt{5}} \log \left| \frac{x-\sqrt{5}}{x+\sqrt{5}} \right| + C$$

$$= x + \frac{1}{4\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + \frac{27}{8\sqrt{5}} \log \left| \frac{x-\sqrt{5}}{x+\sqrt{5}} \right| + C$$

48. Let $I = \int \frac{1}{\sin x + \sin 2x} dx$

$$= \int \frac{1}{\sin x + 2\sin x \cos x} dx = \int \frac{1}{\sin x(1+2\cos x)} dx$$

$$= \int \frac{\sin x}{\sin^2 x(1+2\cos x)} dx$$

Let $u = \cos x \Rightarrow du = -\sin x dx$

Also, $\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$$\therefore I = \int \frac{-1}{(1-u^2)(1+2u)} du$$

$$= \int \frac{-1}{(1+u)(1-u)(1+2u)} du$$

Using partial fractions, we have

$$\frac{-1}{(1+u)(1-u)(1+2u)} = \frac{A}{1+u} + \frac{B}{1-u} + \frac{C}{1+2u}$$

$$\Rightarrow -1 = A(1-u)(1+2u) + B(1+u)(1+2u) + C(1+u)(1-u)$$

Putting $u = 1$, we get $B = -1/6$

Putting $u = -1$, we get $A = 1/2$

$$\text{Put } u = -\frac{1}{2}, \text{ we get } C = \frac{-4}{3}$$

$$\text{So, } \frac{-1}{(1+u)(1-u)(1+2u)} = \frac{1}{2(1+u)} - \frac{1}{6(1-u)} - \frac{4}{3(1+2u)}$$

$$\Rightarrow I = \int \left[\frac{1}{2(1+u)} - \frac{1}{6(1-u)} - \frac{4}{3(1+2u)} \right] du$$

$$= \frac{1}{2} \log(1+u) + \frac{1}{6} \log(1-u) - \frac{4}{3 \times 2} \log(1+2u) + C_1$$

$$= \frac{1}{2} \log(1+\cos x) + \frac{1}{6} \log(1-\cos x) - \frac{2}{3} \log(1+2\cos x) + C_1$$

49. Let $I = \int \frac{x^2}{(x^2+4)(x^2+9)} dx$

Put $x^2 = y$. Then $\frac{x^2}{(x^2+4)(x^2+9)} = \frac{y}{(y+4)(y+9)}$

$$\text{Let } \frac{y}{(y+4)(y+9)} = \frac{A}{y+4} + \frac{B}{y+9} \quad \dots(i)$$

$$\Rightarrow y = A(y+9) + B(y+4) \quad \dots(ii)$$

Putting $y = -4$ and $y = -9$ successively in (ii), we get

$$A = \frac{-4}{5} \text{ and } B = \frac{9}{5}$$

Substituting the values of A and B in (i), we get

$$\frac{y}{(y+4)(y+9)} = \frac{-4/5}{y+4} + \frac{9/5}{y+9}$$

$$\Rightarrow \frac{x^2}{(x^2+4)(x^2+9)} = \frac{-4}{5(x^2+4)} + \frac{9}{5(x^2+9)}$$

$$\begin{aligned}
 \therefore I &= \int \frac{x^2}{(x^2+4)(x^2+9)} dx \\
 &= \frac{-4}{5} \int \frac{1}{(x^2+4)} dx + \frac{9}{5} \int \frac{1}{(x^2+9)} dx \\
 &= \frac{-4}{5} \times \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + \frac{9}{5} \times \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C \\
 &= \frac{-2}{5} \tan^{-1} \left(\frac{x}{2} \right) + \frac{3}{5} \tan^{-1} \left(\frac{x}{3} \right) + C
 \end{aligned}$$

50. Let $I = \int \frac{x}{(x-1)^2(x+2)} dx$ (i)

Using partial fraction, we have

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \quad \dots(ii)$$

$$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \quad \dots(iii)$$

On comparing the coefficients of x^2 , x and constant term in (iii), we get $0 = A + C$, $1 = A + B - 2C$, $0 = -2A + 2B + C$

Solving these, we get

$$A = \frac{2}{9}, B = \frac{1}{3}, C = -\frac{2}{9}$$

From (ii), we get $\frac{x}{(x-1)^2(x+2)} = \frac{2}{9} \cdot \frac{1}{x-1} + \frac{1}{3} \cdot \frac{1}{(x-1)^2} - \frac{2}{9} \cdot \frac{1}{x+2}$

$$\therefore I = \frac{2}{9} \int \frac{1}{x-1} dx + \frac{1}{3} \int \frac{1}{(x-1)^2} dx - \frac{2}{9} \int \frac{1}{x+2} dx$$

$$\Rightarrow I = \frac{2}{9} \log|x-1| - \frac{1}{3} \cdot \frac{1}{x-1} - \frac{2}{9} \log|x+2|$$

$$= \frac{2}{9} \log \left| \frac{x-1}{x+2} \right| - \frac{1}{3} \cdot \frac{1}{x-1} + C_1$$

51. Let $I = \int \frac{x}{(x^2+1)(x-1)} dx$

$$\text{Let } \frac{x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} \quad \dots(i)$$

$$\Rightarrow x = (Ax+B)(x-1) + C(x^2+1) \quad \dots(ii)$$

Comparing coefficients of x^2 , x and constant terms, we get $0 = A + C$, $1 = B - A$, $0 = -B + C$

Solving these, we get

$$A = -\frac{1}{2}, C = \frac{1}{2} \text{ and } B = \frac{1}{2}$$

\therefore From (i), we get

$$\frac{x}{(x^2+1)(x-1)} = \frac{-\frac{1}{2}(x-1)}{x^2+1} + \frac{1}{2} \cdot \frac{1}{x-1}$$

$$= -\frac{1}{2} \cdot \frac{x}{x^2+1} + \frac{1}{2} \cdot \frac{1}{x^2+1} + \frac{1}{2} \cdot \frac{1}{x-1}$$

$$\therefore I = -\frac{1}{4} \int \frac{2x}{x^2+1} dx + \frac{1}{2} \int \frac{dx}{x^2+1} + \frac{1}{2} \int \frac{dx}{x-1}$$

$$\Rightarrow I = -\frac{1}{4} \log|x^2+1| + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \log|x-1| + C_1$$

52. Let $I = \int \frac{x^2}{(x^2+1)(x^2+4)} dx$

Put $x^2 = y$

$$\text{So, } \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

$$\text{Let } \frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4} \quad \dots(i)$$

$$\Rightarrow y = A(y+4) + B(y+1) \quad \dots(ii)$$

Putting $y = -4$ and $y = -1$, successively in (ii), we get

$$A = \frac{-1}{3} \text{ and } B = \frac{4}{3}$$

From (i), we get

$$\frac{y}{(y+1)(y+4)} = \frac{-1}{3(y+1)} + \frac{4}{3(y+4)}$$

$$\Rightarrow \frac{x^2}{(x^2+1)(x^2+4)} = \frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

$$\therefore I = \int \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{-1}{3} \int \frac{dx}{x^2+1} + \frac{4}{3} \int \frac{dx}{x^2+4}$$

$$= \frac{-1}{3} \times \tan^{-1} x + \frac{4}{3} \times \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C$$

$$= \frac{-1}{3} \tan^{-1} x + \frac{2}{3} \tan^{-1} \left(\frac{x}{2} \right) + C$$

53. Let $I = \int \frac{x^3}{x^4+3x^2+2} dx$

Put $x^2 = t \Rightarrow x dx = \frac{1}{2} dt$

$$\therefore I = \frac{1}{2} \int \frac{t}{t^2+3t+2} dt = \frac{1}{2} \int \frac{t}{(t+2)(t+1)} dt \quad \dots(i)$$

$$\text{Let } \frac{t}{(t+2)(t+1)} = \frac{A}{t+2} + \frac{B}{t+1}$$

$$\Rightarrow t = A(t+1) + B(t+2)$$

Put $t = -1, -2$ in it, we get $A = 2, B = -1$

$$\therefore \frac{t}{(t+2)(t+1)} = \frac{2}{t+2} - \frac{1}{t+1} \quad \dots(ii)$$

$$\text{From (i) and (ii), we get } I = \frac{1}{2} \int \left[\frac{2}{t+2} - \frac{1}{t+1} \right] dt$$

$$= \frac{1}{2} [2 \log|t+2| - \log|t+1|] + C = \frac{1}{2} [2 \log|x^2+2| - \log|x^2+1|] + C$$

Concept Applied 

$$\Rightarrow \int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$$

54. Let $I = \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$

$$\text{Let } \frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \quad \dots(i)$$

$$\Rightarrow x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

Put $x = -1, -2, 0$ successively in it, we get

$$B = 1; C = 3; A = -2$$

From (i), we get

$$\frac{x^2+x+1}{(x+1)^2(x+2)} = \frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}$$

Integrating both sides w.r.t. x , we get

$$I = \int \frac{x^2+x+1}{(x+1)^2(x+2)} dx$$

$$= -2 \int \frac{1}{x+1} dx + \int \frac{1}{(x+1)^2} dx + 3 \int \frac{1}{x+2} dx$$

$$= -2 \log|x+1| - \frac{1}{x+1} + 3 \log|x+2| + C_1$$

55. (d) : Let $I = \int e^x \left(\log x + \frac{1}{x} \right) dx$

$$\Rightarrow I = e^x \log x + c \quad (\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + c)$$


56. (d) : Let $I = \int \frac{e^x}{x+1} [1 + (x+1) \log(x+1)] dx$

$$= \int e^x \left[\frac{1}{x+1} + \log(x+1) \right] dx$$

It is of the form $\int e^x [f(x) + f'(x)] dx$,

where $f(x) = \log(x+1)$ and $f'(x) = \frac{1}{x+1}$

So, $I = e^x \log(x+1) + C$

Concept Applied 

$$\Rightarrow \int e^x [f(x) + f'(x)] dx = e^x f(x) + c$$

$$57. \text{ Let } I = \int x^4 \log x \, dx = \log x \cdot \frac{x^5}{5} - \int \frac{1}{x} \cdot \frac{x^5}{5} \, dx$$

$$= \frac{x^5}{5} \log x - \frac{1}{5} \int x^4 \, dx = \frac{1}{5} x^5 \log x - \frac{x^5}{25} + C$$

$$58. \text{ Let } I = \int \frac{\log x - 3}{(\log x)^4} \, dx$$

$$\text{Put } \log x = t \Rightarrow x = e^t \Rightarrow dx = e^t \, dt$$

$$\therefore I = \int \left[\frac{t-3}{t^4} \right] e^t \, dt = \int e^t t^{-3} \, dt - 3 \int t^{-4} e^t \, dt$$

$$= t^{-3} e^t + 3 \int t^{-4} e^t \, dt - 3 \int t^{-4} e^t \, dt + C$$

$$= t^{-3} e^t + C = (\log x)^{-3} x + C$$

$$59. \text{ Let } I = \int \sin^{-1}(2x) \, dx = \int 1 \cdot \sin^{-1}(2x) \, dx$$

$$= \sin^{-1}(2x) x - \int \left(\frac{1}{\sqrt{1-4x^2}} \frac{d}{dx}(2x) \cdot x \right) dx$$

$$= x \sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}} \, dx$$

$$= x \sin^{-1}(2x) + \int \frac{dt}{4\sqrt{t}} \quad (\text{Putting } 1-4x^2 = t \Rightarrow -8x \, dx = dt)$$

$$= x \sin^{-1}(2x) + \frac{2}{4} (t)^{1/2} + C = x \sin^{-1}(2x) + \frac{1}{2} \sqrt{1-4x^2} + C$$

$$60. \text{ Let } I = \int x \cdot \tan^{-1} x \, dx$$

On integrating by parts w.r.t. x , we get

$$I = \tan^{-1} x \int x \, dx - \int \left[\frac{d}{dx}(\tan^{-1} x) \right] x \, dx$$

$$= (\tan^{-1} x) \frac{x^2}{2} - \int \frac{1}{(1+x^2)} \frac{x^2}{2} \, dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{1+x^2} \, dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2} \tan^{-1} x + C$$

$$= \frac{1}{2} (1+x^2) \tan^{-1} x - \frac{x}{2} + C$$

$$61. \text{ Let } I = \int e^x \sin 2x \, dx$$

$$= \left[\sin 2x e^x - 2 \int \cos 2x e^x \, dx \right]$$

$$= \left[e^x \sin 2x - 2 \int e^x \cos 2x \, dx \right]$$

$$= \left[e^x \sin 2x - 2 \left[\cos 2x e^x + 2 \int \sin 2x e^x \, dx \right] \right]$$

$$\Rightarrow I = e^x \sin 2x - 2e^x \cos 2x - 4I + C$$

$$\Rightarrow 5I = e^x \sin 2x - 2e^x \cos 2x + C$$

$$\Rightarrow I = \frac{1}{5} (e^x \sin 2x - 2e^x \cos 2x) + \frac{C}{5}$$

$$= \frac{e^x}{5} (\sin 2x - 2 \cos 2x) + C_1, \text{ where } C_1 = \frac{C}{5} \text{ is an arbitrary constant.}$$

Answer Tips

$$\Rightarrow \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx] + C$$

$$62. \text{ Let } I = \int \sec^3 x \, dx = \int \sec x \cdot \sec^2 x \, dx$$

$$= \sec x \tan x - \int \sec x \tan x \cdot \tan x \, dx$$

(Applying integration by parts)

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - I + \ln |\sec x + \tan x| + C_1$$

$$\Rightarrow 2I = \sec x \tan x + \ln |\sec x + \tan x| + C_1$$

$$\therefore I = \frac{1}{2} (\sec x \tan x) + \frac{1}{2} \ln |\sec x + \tan x| + C \quad \left(\text{Where, } C = \frac{C_1}{2} \right)$$

Key Points

$$\Rightarrow \int \sec x \, dx = \log |\sec x + \tan x| + C$$

$$63. \text{ Let } I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$$

$$= \sin^{-1} x \int \frac{x}{\sqrt{1-x^2}} \, dx - \int \left[\frac{d}{dx}(\sin^{-1} x) \right] \frac{x}{\sqrt{1-x^2}} \, dx$$

(Applying integration by parts)

Firstly, let us evaluate the integral $\int \frac{x}{\sqrt{1-x^2}} \, dx$

$$\text{Put } t = 1 - x^2 \Rightarrow dt = -2x \, dx.$$

$$\therefore \int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{1-x^2}$$

$$\therefore I = \sin^{-1} x (-\sqrt{1-x^2}) - \int \frac{1}{\sqrt{1-x^2}} (-\sqrt{1-x^2}) \, dx$$

$$= -\sqrt{1-x^2} \sin^{-1} x + \int dx = -\sqrt{1-x^2} \sin^{-1} x + x + C$$

$$64. \text{ Let } I = \int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}} \, dx$$

$$= -\int \frac{(-x^2 + 3x - 1)}{\sqrt{1-x^2}} \, dx = -\int \frac{(1-x^2) + 3x - 2}{\sqrt{1-x^2}} \, dx$$

$$\text{i.e., } I = -\int \sqrt{1-x^2} \, dx + \int \frac{-3x+2}{\sqrt{1-x^2}} \, dx$$

$$= -\int \sqrt{1-x^2} \, dx + \frac{3}{2} \int \frac{-2x}{\sqrt{1-x^2}} \, dx + 2 \int \frac{1}{\sqrt{1-x^2}} \, dx$$

$$= -\int \sqrt{1-x^2} \, dx + \frac{3 \times 2}{2} \sqrt{1-x^2} + 2 \sin^{-1} x + C$$

$$= -\left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right] + 3\sqrt{1-x^2} + 2 \sin^{-1} x + C$$

$$= -\frac{x}{2} \sqrt{1-x^2} + \frac{3}{2} \sin^{-1} x + 3\sqrt{1-x^2} + C$$

Commonly Made Mistake

$$\Rightarrow \text{Remember the formula for } \int \sqrt{a^2 - x^2} \, dx \text{ and } \int \frac{1}{\sqrt{a^2 - x^2}} \, dx.$$

$$65. \text{ Let } I = \int \frac{\log x}{(x+1)^2} dx = \int (x+1)^{-2} \cdot \log x dx$$

On integrating by parts, taking $\log x$ as first function, we have

$$\begin{aligned} I &= \frac{(x+1)^{-1}}{-1} \cdot \log x - \int \frac{(x+1)^{-1}}{-1} \cdot \frac{1}{x} dx \\ &= \frac{-\log x}{x+1} + \int \frac{dx}{x(x+1)} = \frac{-\log x}{x+1} + \int \left[\frac{1}{x} - \frac{1}{x+1} \right] dx \\ &= \frac{-\log x}{x+1} + \log x - \log(x+1) + C \\ &= \frac{-\log x}{x+1} + \log \left(\frac{x}{x+1} \right) + C \end{aligned}$$

$$66. \text{ Let } I = \int e^{2x} \sin(3x+1) dx$$

On integrating by parts, taking e^x as first function, we have

$$\begin{aligned} &= e^{2x} \int \sin(3x+1) dx - \int \left(\frac{d(e^{2x})}{dx} \cdot \int \sin(3x+1) dx \right) dx \\ &= e^{2x} \left[\frac{-\cos(3x+1)}{3} \right] - \int 2e^{2x} \cdot \left[\frac{-\cos(3x+1)}{3} \right] dx \\ &= \frac{-e^{2x} \cos(3x+1)}{3} + \frac{2}{3} \int e^{2x} \cos(3x+1) dx \\ &= \frac{-e^{2x} \cos(3x+1)}{3} + \frac{2}{3} \left[\int e^{2x} \cos(3x+1) dx \right. \\ &\quad \left. - \int \left(\frac{d}{dx}(e^{2x}) \cdot \int \cos(3x+1) dx \right) dx \right] \\ &= \frac{-e^{2x} \cos(3x+1)}{3} + \frac{2}{9} e^{2x} \sin(3x+1) - \frac{4}{9} \int e^{2x} \sin(3x+1) dx \\ &\Rightarrow I + \frac{4}{9} I = \frac{-e^{2x} \cos(3x+1)}{3} + \frac{2}{9} e^{2x} \sin(3x+1) + C_1 \\ &\Rightarrow \frac{13I}{9} = \frac{-e^{2x} \cos(3x+1)}{3} + \frac{2}{9} e^{2x} \sin(3x+1) + C_1 \\ &\Rightarrow I = \frac{9}{13} \left[\frac{-e^{2x} \cos(3x+1)}{3} + \frac{2}{9} e^{2x} \sin(3x+1) + C_1 \right] \\ &= \frac{9}{13} e^{2x} \left[\frac{2 \sin(3x+1) - 3 \cos(3x+1)}{9} \right] + \frac{9}{13} C_1 \\ &= \frac{1}{13} e^{2x} [2 \sin(3x+1) - 3 \cos(3x+1)] + C \quad \left(\text{Where, } C = \frac{9C_1}{13} \right) \end{aligned}$$

$$67. \text{ Let } I = \int \frac{x^2+1}{(x+1)^2} e^x dx$$

$$\begin{aligned} &= \int e^x \cdot \left[\frac{(x^2-1)+2}{(x+1)^2} \right] dx \\ &= \int e^x \cdot \left[\frac{x^2-1}{(x+1)^2} + \frac{2}{(x+1)^2} \right] dx \\ &= \int e^x \cdot \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx \end{aligned}$$

$$\text{Take } f(x) = \frac{x-1}{x+1}$$

$$\Rightarrow f'(x) = \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

By using the formula, we get

$$I = \int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

$$I = e^x \left[\frac{x-1}{x+1} \right] + C$$

$$68. \text{ Let } I = \int \frac{x \cos^{-1} x}{\sqrt{1-x^2}} dx$$

Put $\cos^{-1} x = \theta \Rightarrow x = \cos \theta \Rightarrow dx = -\sin \theta d\theta$

$$\Rightarrow I = \int \frac{\cos \theta (\theta)}{\sqrt{1-\cos^2 \theta}} (-\sin \theta) d\theta$$

$$\Rightarrow I = -\int \theta \cos \theta d\theta$$

$$\Rightarrow -I = \theta \int \cos \theta d\theta - \int \left(\frac{d}{d\theta} \theta \int \cos \theta d\theta \right) d\theta$$

(Applying integration by parts)

$$\Rightarrow -I = \theta \sin \theta - \int \sin \theta d\theta$$

$$\Rightarrow -I = \theta \sin \theta + \cos \theta + C$$

$$\Rightarrow I = -[\theta \sqrt{1-x^2} + \cos \theta] + C$$

$$\Rightarrow I = -[\sqrt{1-x^2} \cos^{-1} x + x] + C$$

Key Points

→ The value of $\sin(-\theta)$ is -ve and value of $\cos(-\theta)$ is +ve.

$$69. \text{ Let } I = \int \frac{\sqrt{x^2+1} [\log(x^2+1) - 2 \log x]}{x^4} dx$$

$$= \int \frac{\sqrt{x^2+1} \log \left(\frac{x^2+1}{x^2} \right)}{x^4} dx$$

$$\Rightarrow 2x dx = \frac{-1}{(t-1)^2} dt \Rightarrow dx = -\frac{1}{2x} \cdot \frac{1}{(t-1)^2} dt$$

$$= -\frac{1}{2} \cdot \sqrt{t-1} \cdot \frac{1}{(t-1)^2} dt = -\frac{dt}{2(t-1)^{3/2}}$$

$$\text{Also, } \sqrt{x^2+1} = \sqrt{\frac{1}{t-1}+1} = \sqrt{\frac{t}{t-1}}$$

$$\therefore I = \int \sqrt{\frac{t}{t-1}} \cdot \log t \cdot \frac{1}{1/(t-1)^2} \times \frac{-dt}{2(t-1)^{3/2}} = -\frac{1}{2} \int \sqrt{t} \cdot \log t dt$$

On integrating by parts, taking $\log t$ as first function, we have

$$= -\frac{1}{2} \left[\frac{t^{3/2}}{3/2} \cdot \log t - \int \frac{t^{3/2}}{3/2} \cdot \frac{1}{t} dt \right] + C$$

$$= -\frac{1}{3} \left[t^{3/2} \log t - \int t^{1/2} dt \right] + C$$

$$= -\frac{1}{3} \left[t^{3/2} \log t - \frac{2}{3} t^{3/2} \right] + C$$

$$= -\frac{1}{3} \left[\left(\frac{x^2+1}{x^2} \right)^{3/2} \log \left(\frac{x^2+1}{x^2} \right) - \frac{2}{3} \left(\frac{x^2+1}{x^2} \right)^{3/2} \right] + C$$

$$70. \text{ Let } I = \int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} dx, x \in [0,1]$$

We know that $\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x} = \frac{\pi}{2}$

...(i)

$$\Rightarrow \sin^{-1} \sqrt{x} = \frac{\pi}{2} - \cos^{-1} \sqrt{x}$$

$$\begin{aligned} \therefore I &= \int \frac{\frac{\pi}{2} - 2\cos^{-1} \sqrt{x}}{\pi/2} dx \\ &= \int 1 \cdot dx - \frac{4}{\pi} \int 1 \cdot \cos^{-1} \sqrt{x} dx \\ &= x - \frac{4}{\pi} \left[x \cdot \cos^{-1} \sqrt{x} - \int x \cdot \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} dx \right] + C \\ &= x - \frac{4}{\pi} \left[x \cos^{-1} \sqrt{x} + \frac{1}{2} \int \frac{\sqrt{x}}{1-x} dx \right] + C \end{aligned}$$

$$\text{Put } x = \sin^2 \theta \Rightarrow dx = 2 \sin \theta \cos \theta d\theta$$

$$\begin{aligned} \therefore I &= x - \frac{4}{\pi} x \cos^{-1} \sqrt{x} - \frac{2}{\pi} \int \frac{\sin^2 \theta}{1 - \sin^2 \theta} \cdot 2 \sin \theta \cos \theta d\theta + C \\ &= x - \frac{4}{\pi} x \cos^{-1} \sqrt{x} - \frac{2}{\pi} \int \frac{\sin \theta}{\cos \theta} \cdot 2 \sin \theta \cos \theta d\theta + C \\ &= x - \frac{4}{\pi} x \cos^{-1} \sqrt{x} - \frac{2}{\pi} \int (1 - \cos 2\theta) d\theta + C \\ &= x - \frac{4}{\pi} x \cos^{-1} \sqrt{x} - \frac{2}{\pi} \left[\theta - \frac{\sin 2\theta}{2} \right] + C \\ &= x - \frac{4}{\pi} x \cos^{-1} \sqrt{x} - \frac{2}{\pi} \left[\theta - \sin \theta \cos \theta \right] + C \\ &= x - \frac{4}{\pi} x \cos^{-1} \sqrt{x} - \frac{2}{\pi} \left[\sin^{-1} \sqrt{x} - \sqrt{x} \sqrt{1-x} \right] + C \end{aligned}$$

Concept Applied

$$\begin{aligned} \Rightarrow \cos 2\theta &= 1 - 2\sin^2 \theta \\ \Rightarrow 2\sin^2 \theta &= 1 - \cos 2\theta \end{aligned}$$

$$\begin{aligned} 71. (d): \text{ Let } I &= \int_{-1}^1 \frac{|x-2|}{x-2} dx \\ &= \int_{-1}^1 \frac{-(x-2)}{x-2} dx = \int_{-1}^1 -1 \cdot dx = [-x]_{-1}^1 = -[1 - (-1)] = -2 \end{aligned}$$

$$\begin{aligned} 72. (a): \text{ Let } I &= \int_0^4 (e^{2x} + x) dx = \left[\frac{e^{2x}}{2} + \frac{x^2}{2} \right]_0^4 \\ &= \frac{e^8}{2} + \frac{16}{2} - \frac{e^0}{2} - 0 = \frac{e^8}{2} + \frac{16}{2} - \frac{1}{2} = \frac{e^8 + 15}{2} \end{aligned}$$

$$\begin{aligned} 73. (a): \text{ Let } I &= \int_0^{\pi/8} \tan^2(2x) dx = \int_0^{\pi/8} (\sec^2(2x) - 1) dx \\ &= \left(\frac{1}{2} \tan 2x - x \right)_0^{\pi/8} = \frac{1}{2} \tan 2\left(\frac{\pi}{8}\right) - \frac{\pi}{8} = \frac{1}{2} \tan \frac{\pi}{4} - \frac{\pi}{8} \\ &= \frac{1}{2} - \frac{\pi}{8} = \frac{4 - \pi}{8} \end{aligned}$$

$$\begin{aligned} 74. (d): \text{ Let } I &= \int_{-\pi/4}^{\pi/4} \sec^2 x dx = [\tan x]_{-\pi/4}^{\pi/4} \\ &= \tan \frac{\pi}{4} - \tan \left(-\frac{\pi}{4}\right) = 1 + 1 = 2 \end{aligned}$$

$$75. \text{ We have, } \int_2^3 3^x dx = \left[\frac{3^x}{\log 3} \right]_2^3 = \frac{3^3 - 3^2}{\log 3} = \frac{18}{\log 3}$$

$$76. \int \frac{dx}{9+x^2} = \int \frac{dx}{x^2+3^2} = \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right] = F(x)$$

\therefore By second fundamental theorem of integral calculus, we have

$$\int_0^3 \frac{dx}{9+x^2} = F(3) - F(0) = \frac{1}{3} [\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{3} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{12}$$

Answer Tips

$$\Rightarrow \int \frac{1}{a^2 + b^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right)$$

$$77. \text{ Let } I = \int_0^{\pi/2} e^x (\sin x - \cos x) dx$$

$$= \int_0^{\pi/2} e^x \sin x dx - \int_0^{\pi/2} e^x \cos x dx$$

On integrating II integral by parts, we get

$$\begin{aligned} I &= \int_0^{\pi/2} e^x \sin x dx - \left[\{e^x \cos x\}_0^{\pi/2} - \int_0^{\pi/2} e^x (-\sin x) dx \right] \\ &= \int_0^{\pi/2} e^x \sin x dx - [e^{\pi/2} \cdot 0 - e^0 \cdot 1] - \int_0^{\pi/2} e^x \sin x dx = 1 \end{aligned}$$

78. By the first fundamental theorem of integral calculus

$$f'(x) = \frac{d}{dx} \int_0^x t \sin t dt = x \sin x.$$

$$79. \text{ Given, } \int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$$

$$\Rightarrow \int_0^a \frac{1}{x^2+2^2} dx = \frac{\pi}{8} \Rightarrow \left[\frac{1}{2} \tan^{-1} \frac{x}{2} \right]_0^a = \frac{\pi}{8}$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{a}{2} = \frac{\pi}{8} \Rightarrow \tan^{-1} \frac{a}{2} = \frac{\pi}{4}$$

$$\Rightarrow \frac{a}{2} = \tan \frac{\pi}{4} = 1 \Rightarrow a = 2$$

$$80. \text{ We have, } \int_0^{\pi/4} \tan x dx = [\log |\sec x|]_0^{\pi/4}$$

$$= \log \left| \sec \frac{\pi}{4} \right| - \log |\sec 0| = \log |\sqrt{2}| - \log |1|$$

$$\therefore \int_0^{\pi/4} \tan x dx = \frac{1}{2} \log 2$$

$$81. \text{ We have, } \int \sin 2x dx = -\frac{1}{2} [\cos 2x] = F(x)$$

\therefore By second fundamental theorem of integral calculus,

$$\text{we have } \int_0^{\pi/4} \sin 2x dx = F\left(\frac{\pi}{4}\right) - F(0)$$

$$\Rightarrow \int_0^{\pi/4} (\sin 2x) dx = -\frac{1}{2} [\cos(\pi/2) - \cos(0)] = \frac{1}{2}$$

$$82. \text{ We have, } \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = [\sin^{-1} x]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$83. \text{ Here, } \int_1^2 \frac{x^3-1}{x^2} dx = \int_1^2 (x-x^{-2}) dx$$

$$= \left[\frac{x^2}{2} - \frac{x^{-1}}{-1} \right]_1^2 = \left[\frac{x^2}{2} + \frac{1}{x} \right]_1^2$$

$$= \left[\frac{4}{2} + \frac{1}{2} \right] - \left[\frac{1}{2} + 1 \right] = \frac{5}{2} - \frac{3}{2} = 1$$

$$84. \int_0^1 x^2 e^x dx = [x^2 e^x]_0^1 - 2 \int_0^1 x e^x dx$$

$$= e - 2[xe^x - e^x]_0^1 = e - 2(0 - (-1)) = e - 2$$

$$85. \text{ Let } I = \int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2} \right] e^{2x} dx$$

$$\text{Putting } 2x = y \Rightarrow 2dx = dy$$

$$\text{When } x \rightarrow 1, \text{ then } y \rightarrow 2$$

$$\text{and when } x \rightarrow 2, \text{ then } y \rightarrow 4$$

$$\therefore I = \frac{1}{2} \int_2^4 \left[\frac{2}{y} - \frac{2}{y^2} \right] e^y dy = \int_2^4 \left[\frac{1}{y} - \frac{1}{y^2} \right] e^y dy$$

$$\Rightarrow I = \left[e^y \cdot \frac{1}{y} \right]_2^4 = \frac{1}{4} e^4 - \frac{1}{2} e^2 = \frac{e^2}{2} \left(\frac{e^2}{2} - 1 \right)$$

$$86. \text{ Let } I = \int_0^1 \tan^{-1} \left(\frac{1-2x}{1+x-x^2} \right) dx$$

$$= \int_0^1 \tan^{-1} \left[\frac{(1-x)-x}{1+x(1-x)} \right] dx$$

$$I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1} x] dx \quad \dots(i)$$

$$I = \int_0^1 [\tan^{-1} x - \tan^{-1}(1-x)] dx \quad \dots(ii)$$

$$\left[\text{Using properly } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

Adding (i) and (ii), we get

$$2I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1} x + \tan^{-1} x - \tan^{-1}(1-x)] dx = 0$$

$$\Rightarrow I = 0$$

$$87. \text{ Let } I = \int_0^{\pi/2} x^2 \sin x dx$$

On integrating by parts, we have

$$I = [x^2(-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} 2x(-\cos x) dx$$

$$= -\frac{\pi^2}{4} \cdot 0 + 0 + 2 \int_0^{\pi/2} x \cos x dx = 2 \int_0^{\pi/2} x \cos x dx$$

Again integrating by parts, we have

$$I = 2 \left[x \sin x \Big|_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \sin x dx \right]$$

$$= 2 \left[\frac{\pi}{2} \cdot 1 - 0 - [-\cos x]_0^{\pi/2} \right] = 2 \left[\frac{\pi}{2} + (0-1) \right] = \pi - 2$$

$$88. \text{ Let } I = \int_2^4 \frac{x}{x^2+1} dx$$

$$\text{Put } x^2 + 1 = t \Rightarrow x dx = \frac{1}{2} dt$$

When $x = 2$, then $t = 5$ and when $x = 4$, then $t = 17$

$$\therefore I = \frac{1}{2} \int_5^{17} \frac{dt}{t} = \frac{1}{2} [\log t]_5^{17} = \frac{1}{2} [\log 17 - \log 5] = \frac{1}{2} \log \left(\frac{17}{5} \right)$$

$$89. \text{ Let } I = \int_e^{e^2} \frac{dx}{x \log x}$$

$$\text{Put } \log x = t \Rightarrow \frac{1}{x} dx = dt$$

When $x = e$, then $t = \log e = 1$

and when $x = e^2$, then $t = \log e^2 = 2 \log e = 2$

$$\therefore I = \int_1^2 \frac{dt}{t} = [\log t]_1^2 = \log 2 - \log 1 = \log 2$$

$$90. \text{ Let } I = \int_0^1 x e^{x^2} dx$$

$$\text{Put } x^2 = t \Rightarrow x dx = \frac{1}{2} dt$$

When $x = 0$, then $t = 0$ and when $x = 1$, then $t = 1$.

$$\therefore I = \frac{1}{2} \int_0^1 e^t dt \Rightarrow I = \frac{1}{2} [e^t]_0^1 = \frac{1}{2} (e-1)$$

$$91. \text{ Let } I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$$

$$\text{Put } \tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$$

When $x = 0$, then $t = 0$ and when $x = 1$, then $t = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} t dt = \left[\frac{1}{2} t^2 \right]_0^{\pi/4} = \frac{1}{2} \left[\left(\frac{\pi}{4} \right)^2 - 0 \right] = \frac{\pi^2}{32}$$

$$92. \text{ Let } I = \int_{-\pi/4}^0 \frac{(1+\tan x)}{(1-\tan x)} dx = \int_{-\pi/4}^0 \frac{\left(1 + \frac{\sin x}{\cos x} \right)}{\left(1 - \frac{\sin x}{\cos x} \right)} dx$$

$$= \int_{-\pi/4}^0 \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

$$\text{Put } \cos x - \sin x = t \Rightarrow -(\sin x + \cos x) dx = dt$$

When $x=0$, then $t=1$, when $x=\frac{-\pi}{4}$, then $t=\sqrt{2}$

$$\therefore I = \int_{\sqrt{2}}^1 -\frac{dt}{t} = \int_1^{\sqrt{2}} \frac{dt}{t} = [\log t]_1^{\sqrt{2}} = \log \sqrt{2} - \log 1 = \frac{1}{2} \log 2$$

Concept Applied

$$\Rightarrow \int_b^a f(x) dx = -\int_a^b f(x) dx$$

93. Let $I = \int_{\pi/4}^{\pi/2} e^{2x} \left(\frac{1-\sin 2x}{1-\cos 2x} \right) dx$

Put $2x=t \Rightarrow dx = \frac{1}{2} dt$

When $x = \frac{\pi}{4}$, $t = \frac{\pi}{2}$; When $x = \frac{\pi}{2}$, $t = \pi$

$$\therefore I = \frac{1}{2} \int_{\pi/2}^{\pi} e^t \left(\frac{1-\sin t}{1-\cos t} \right) dt$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} e^t \left(\frac{1-2\sin t/2\cos t/2}{2\sin^2 t/2} \right) dt$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} e^t \left(\frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} - \cot \frac{t}{2} \right) dt$$

$$= \frac{1}{2} \int_{\pi/2}^{\pi} e^t \left(-\cot \frac{t}{2} + \frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} \right) dt$$

$$\Rightarrow I = \left[\frac{1}{2} e^t \left(-\cot \frac{t}{2} \right) \right]_{\pi/2}^{\pi} \quad (\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + C)$$

$$\Rightarrow I = \frac{1}{2} \left[e^{\pi} \left(-\cot \frac{\pi}{2} \right) - e^{\pi/2} \left(-\cot \frac{\pi}{4} \right) \right] = \frac{1}{2} (0 + e^{\pi/2}) = \frac{e^{\pi/2}}{2}$$

94. Let $I = \int_0^{\pi/2} \sqrt{\sin x} \cos^5 x dx$

$$\Rightarrow I = \int_0^{\pi/2} \sqrt{\sin x} (\cos^2 x)^2 \cos x dx$$

$$\Rightarrow I = \int_0^{\pi/2} \sqrt{\sin x} (1-\sin^2 x)^2 \cos x dx$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

When $x=0$, $t = \sin 0 = 0$

When $x = \pi/2$, $t = \sin \pi/2 = 1$

$$\therefore I = \int_0^1 \sqrt{t} (1-t^2)^2 dt = \int_0^1 \sqrt{t} (1+t^4-2t^2) dt$$

$$= \int_0^1 (\sqrt{t} + t^{9/2} - 2t^{5/2}) dt = \left[\frac{t^{3/2}}{3/2} + \frac{t^{11/2}}{11/2} - \frac{2t^{7/2}}{7/2} \right]_0^1$$

$$= \frac{2}{3} + \frac{2}{11} - \frac{4}{7} = \frac{154+42-132}{231} = \frac{64}{231}$$

95. Let $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}} dx = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(1-\sin 2x)}} dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}} dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x) dx = dt$

When $x = \frac{\pi}{3}$, then $t = \frac{\sqrt{3}-1}{2} = \alpha$

and when, $x = \frac{\pi}{6}$, then $t = \frac{1-\sqrt{3}}{2} = -\alpha$

$$\therefore I = \int_{-\alpha}^{\alpha} \frac{dt}{\sqrt{1-t^2}} = [\sin^{-1} t]_{-\alpha}^{\alpha} = 2\sin^{-1} \alpha = 2\sin^{-1} \left(\frac{\sqrt{3}-1}{2} \right)$$

96. Let $I = \int_0^{\pi} e^{2x} \cdot \sin \left(\frac{\pi}{4} + x \right) dx$

Put $\frac{\pi}{4} + x = t \Rightarrow x = t - \frac{\pi}{4} \Rightarrow dx = dt$

When $x=0$, $t = \frac{\pi}{4}$ and when $x = \pi$, $t = \frac{5\pi}{4}$

$$\therefore I = \int_{\pi/4}^{5\pi/4} e^{2(t-\pi/4)} \sin t dt = e^{-\pi/2} \int_{\pi/4}^{5\pi/4} e^{2t} \sin t dt$$

$$= e^{-\pi/2} \left[\left(\sin t \frac{e^{2t}}{2} \right)_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4} \cos t \frac{e^{2t}}{2} dt \right]$$

$$= e^{-\pi/2} \left[\frac{1}{2} \left(e^{5\pi/2} \sin \frac{5\pi}{4} - e^{\pi/2} \sin \frac{\pi}{4} \right) \right.$$

$$\left. - \left(\frac{e^{2t}}{4} \cos t \right)_{\pi/4}^{5\pi/4} - \int_{\pi/4}^{5\pi/4} \frac{e^{2t}}{4} \sin t dt \right]$$

$$= e^{-\pi/2} \left[\frac{1}{2} \left(-\frac{1}{\sqrt{2}} e^{5\pi/2} - \frac{1}{\sqrt{2}} e^{\pi/2} \right) \right.$$

$$\left. - \frac{1}{4} \left(-\frac{1}{\sqrt{2}} e^{5\pi/2} - \frac{1}{\sqrt{2}} e^{\pi/2} \right) \right] - \frac{I}{4}$$

$$\Rightarrow I + \frac{1}{4} I = -\frac{1}{2\sqrt{2}} [e^{2\pi} + 1] + \frac{1}{4\sqrt{2}} [e^{2\pi} + 1]$$

$$\Rightarrow \frac{5}{4} I = \frac{(e^{2\pi} + 1)}{2\sqrt{2}} \left[\frac{1}{2} - 1 \right] = -\frac{1}{4\sqrt{2}} [e^{2\pi} + 1] \Rightarrow I = \frac{-1}{5\sqrt{2}} (1 + e^{2\pi})$$

97. Let $I = \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2\sin 2x}} = \int_0^{\pi/4} \frac{dx}{\cos^3 x \sqrt{2} \cdot 2\sin x \cos x}$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^{(3+1)} x \cdot \sin^{\frac{1}{2}} x} = \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^2 x \cdot \tan^{\frac{1}{2}} x \cdot \cos^2 x}$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{dx}{\cos^4 x \sqrt{\tan x}} = \frac{1}{2} \int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x}} dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

When $x=0$, then $t=0$ and when $x = \frac{\pi}{4}$, then $t=1$

$$\therefore I = \frac{1}{2} \int_0^1 \frac{(1+t^2) dt}{\sqrt{t}} = \frac{1}{2} \int_0^1 (t^{-\frac{1}{2}} + t^{\frac{3}{2}}) dt$$

$$= \frac{1}{2} \left[\frac{t^{1/2}}{1/2} + \frac{t^{5/2}}{5/2} \right]_0^1 = \frac{1}{2} \left[2 + \frac{2}{5} \right] = 1 + \frac{1}{5} = \frac{6}{5}$$

98. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9 \sin 2x} dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x) dx = dt$

and $1 - 2 \sin x \cos x = t^2 \Rightarrow 1 - \sin 2x = t^2$

When $x = \frac{\pi}{4}$, $t = \sin \frac{\pi}{4} - \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$

When $x = 0$, $t = \sin 0 - \cos 0 = -1$

$\therefore \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9 \sin 2x} dx = \int_{-1}^0 \frac{dt}{16 + 9(1-t^2)}$

$= \int_{-1}^0 \frac{dt}{25 - 9t^2} = \frac{1}{9} \int_{-1}^0 \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2} = \frac{1}{9} \cdot \frac{1}{2 \times \frac{5}{3}} \left[\log \left| \frac{\frac{5}{3} + t}{\frac{5}{3} - t} \right| \right]_{-1}^0$

$= \frac{1}{30} [\log 1 - (\log 1 - \log 4)] = \frac{1}{30} \log 4$

Key Points

$\Rightarrow \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$

99. Let $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x) dx = dt$

Also, $1 - 2 \sin x \cos x = t^2 \Rightarrow 1 - \sin 2x = t^2$

When $x = \frac{\pi}{4}$, $t = 0$ and when $x = 0$, $t = -1$

$\therefore I = \int_{-1}^0 \frac{dt}{9 + 16(1-t^2)}$

$= \int_{-1}^0 \frac{dt}{25 - 16t^2} = \frac{1}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2}$

$= \frac{1}{16} \cdot \frac{4}{2 \times 5} \left[\log \left| \frac{5/4 + t}{5/4 - t} \right| \right]_{-1}^0 = \frac{1}{40} \left[\log 1 - \log \left(\frac{1}{9} \right) \right]$

$= \frac{1}{40} [\log 1 - \log 1 + \log 9] = \frac{1}{40} \log 9$

100. (a): Let $I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}} dx$... (i)

$= \int_2^8 \frac{\sqrt{10-(10-x)}}{\sqrt{10-x} + \sqrt{10-(10-x)}} dx \left(\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right)$

$= \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x} + \sqrt{x}} dx$... (ii)

Adding (i) and (ii), we get

$2I = \int_2^8 \frac{\sqrt{10-x} + \sqrt{x}}{\sqrt{x} + \sqrt{10-x}} dx = \int_2^8 1 dx = [x]_2^8$

$\Rightarrow I = \frac{1}{2}(8-2) = \frac{6}{2} = 3$

Hence, both assertion and reason are true and reason is the correct explanation of assertion.

101. Let $I = \int_0^{\pi/2} \frac{1}{1 + \cot^{5/2} x} dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^{5/2} x}{\sin^{5/2} x + \cos^{5/2} x} dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^{5/2} \left(\frac{\pi}{2} - x \right)}{\sin^{5/2} \left(\frac{\pi}{2} - x \right) + \cos^{5/2} \left(\frac{\pi}{2} - x \right)} dx$... (i)

$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$

$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^{5/2} x}{\cos^{5/2} x + \sin^{5/2} x} dx$... (ii)

Adding (i) and (ii), we get

$2I = \int_0^{\pi/2} \frac{\sin^{5/2} x + \cos^{5/2} x}{\sin^{5/2} x + \cos^{5/2} x} dx$

$\Rightarrow 2I = \int_0^{\pi/2} 1 dx = (x)_0^{\pi/2} = \frac{\pi}{2} \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

102. Let $I = \int_1^3 |2x-1| dx$

$= \int_1^3 (2x-1) dx = \left[\frac{2x^2}{2} - x \right]_1^3$

$= [(3^2 - 3) - (1^2 - 1)] = [(9 - 3) - (1 - 1)] = 6$

103. Let $I = \int_{-2}^2 |x| dx$

$\therefore I = \int_{-2}^0 (-x) dx + \int_0^2 x dx = \left[-\frac{x^2}{2} \right]_{-2}^0 + \left[\frac{x^2}{2} \right]_0^2$
 $= 2 + 2 = 4$

104. Let $I = \int_1^4 |x-5| dx$

$= -\int_1^4 (x-5) dx = \left[-\frac{x^2}{2} + 5x \right]_1^4$
 $= -\frac{16}{2} + 5(4) + \frac{1}{2} - 5 = -8 + 20 - 5 + \frac{1}{2} = 7 + \frac{1}{2} = \frac{15}{2}$

105. Let $I = \int_0^{\pi/2} \log \left(\frac{4+3\sin x}{4+3\cos x} \right) dx$... (i)

$\Rightarrow I = \int_0^{\pi/2} \log \left(\frac{4+3\sin \left(\frac{\pi}{2} - x \right)}{4+3\cos \left(\frac{\pi}{2} - x \right)} \right) dx$

$\left[\text{Using property } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$

$\Rightarrow I = \int_0^{\pi/2} \log \left(\frac{4+3\cos x}{4+3\sin x} \right) dx$... (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) dx + \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right) \cdot \left(\frac{4+3\cos x}{4+3\sin x}\right) dx$$

$$= \int_0^{\pi/2} \log 1 dx = 0 \Rightarrow I = 0$$

106. Let $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}} dx$

$$\Rightarrow I = \int_0^{2\pi} \frac{1}{1+e^{\sin(2\pi-x)}} dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^{2\pi} \frac{1}{1+e^{-\sin x}} dx$$

$$= \int_0^{2\pi} \frac{e^{\sin x}}{e^{\sin x} + 1} dx$$

Adding (i) and (ii), we get

$$2I = \int_0^{2\pi} \left(\frac{1+e^{\sin x}}{1+e^{\sin x}} \right) dx = \int_0^{2\pi} 1 \cdot dx$$

$$\Rightarrow I = \frac{1}{2} [x]_0^{2\pi} = \frac{1}{2} \times 2\pi = \pi$$

107. Let $I = \int_{-2}^2 \frac{x^2}{1+5^x} dx$

Using property: $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$\therefore I = \int_{-2}^2 \frac{(-2+2-x)^2}{1+5^{-2-x}} dx \Rightarrow I = \int_{-2}^2 \frac{x^2}{1+5^{-x}} dx$$

$$\Rightarrow I = \int_{-2}^2 \frac{5^x \cdot x^2}{5^x + 1} dx$$

Adding (i) and (ii), we get

$$2I = \int_{-2}^2 \frac{5^x x^2 + x^2}{5^x + 1} dx$$

$$\Rightarrow 2I = \int_{-2}^2 x^2 dx = \left[\frac{x^3}{3} \right]_{-2}^2 \Rightarrow I = \frac{1}{6} (8+8) = \frac{16}{6} = \frac{8}{3}$$

108. Let $I = \int_1^4 (|x| + |3-x|) dx$

$$|x| + |3-x| = \begin{cases} x+3-x, & 1 \leq x < 2 \\ x+3-x, & 2 \leq x < 3 \\ 2x-3, & 3 \leq x < 4 \end{cases}$$

$$\therefore I = \int_1^3 3 dx + \int_3^4 (2x-3) dx = [3x]_1^3 + \left[\frac{2x^2}{2} - 3x \right]_3^4$$

$$= (9-3) + (16-12-9+9)$$

$$= 6+4 = 10$$

109. Let $I = \int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} dx$... (i)

$$I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{4-x} + \sqrt{x}} dx \quad \dots \text{(ii)} \quad \left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

Adding (i) and (ii), we get

$$2I = \int_1^3 \frac{\sqrt{x} + \sqrt{4-x}}{\sqrt{x} + \sqrt{4-x}} dx$$

$$\dots \text{(i)} \Rightarrow 2I = \int_1^3 1 dx \Rightarrow 2I = [x]_1^3$$

$$\Rightarrow 2I = 2 \Rightarrow I = 1$$

110. Let $I = \int_0^{\pi} \frac{x}{9\sin^2 x + 16\cos^2 x} dx$... (i)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi-x) dx}{9\sin^2(\pi-x) + 16\cos^2(\pi-x)}$$

$$\dots \text{(ii)} \quad \left(\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi-x) dx}{9\sin^2 x + 16\cos^2 x} \quad \dots \text{(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi dx}{9\sin^2 x + 16\cos^2 x}$$

Consider $f(x) = \frac{1}{9\sin^2 x + 16\cos^2 x}$

$$\dots \text{(i)} \quad f(\pi-x) = \frac{1}{9\sin^2(\pi-x) + 16\cos^2(\pi-x)}$$

$$= \frac{1}{9\sin^2 x + 16\cos^2 x} = f(x)$$

$$\therefore I = \pi \int_0^{\pi/2} \frac{dx}{9\sin^2 x + 16\cos^2 x}$$

$$\dots \text{(ii)} \quad \left(\text{Using } \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x) \right)$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{9\tan^2 x + 16} \quad \left(\text{Dividing } N' \text{ \& } D' \text{ by } \cos^2 x \right)$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

When $x = 0, t = 0; x = \pi/2, t = \infty$

$$\Rightarrow I = \pi \int_0^{\infty} \frac{dt}{9t^2 + 16} = \frac{\pi}{9} \int_0^{\infty} \frac{dt}{t^2 + \frac{16}{9}}$$

$$= \frac{\pi}{9} \cdot \frac{3}{4} \left[\tan^{-1} \frac{3t}{4} \right]_0^{\infty} = \frac{\pi}{12} (\tan^{-1} \infty - \tan^{-1} 0) = \frac{\pi^2}{24}$$

111. Let $I = \int_{-1}^2 |x^3 - x| dx = \int_{-1}^2 |x(x-1)(x+1)| dx$

$$= \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx + \int_1^2 (x^3 - x) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 - \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_1^2 = \frac{-3}{4} + \frac{3}{2} + 2 = \frac{11}{4}$$

112. In the R.H.S. integral, put $(a-x) = t$, so that $dx = -dt$.

Now, when $x = 0$, then $t = a$

and when $x = a$, then $t = 0$

$$\therefore \int_0^a f(a-x) dx = -\int_a^0 f(t) dt = \int_0^a f(t) dt = \int_0^a f(x) dx$$

$$\text{Hence, } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$\text{Let } I = \int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\pi-x) \sin(\pi-x)}{1 + [\cos(\pi-x)]^2} dx$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{(\pi-x) \sin x}{1 + \cos^2 x} dx$$

Adding (ii) and (iii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\pi \sin x}{1 + \cos^2 x} dx$$

Put $\cos x = t \Rightarrow -\sin x dx = dt$

When $x = 0$, then $t = 1$ and when $x = \frac{\pi}{2}$, then $t = 0$

$$\therefore 2I = \int_1^0 \frac{-\pi dt}{1+t^2} \Rightarrow I = \frac{\pi}{2} \int_0^1 \frac{dt}{1+t^2}$$

$$\therefore I = \frac{\pi}{2} [\tan^{-1} t]_0^1 = \frac{\pi}{2} [\tan^{-1}(1) - \tan^{-1}(0)]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi^2}{8}$$

113. In the R.H.S. integral, put $(a-x) = t$, so that $dx = -dt$.

Now, when $x = 0$, then $t = a$

and when $x = a$, then $t = 0$

$$\therefore \int_0^a f(a-x) dx = -\int_a^0 f(t) dt = \int_0^a f(t) dt = \int_0^a f(x) dx$$

$$\text{Hence, } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$\text{Now, } \int_0^1 x^2 (1-x)^n dx = \int_0^1 (1-x)^2 (1-(1-x))^n dx \quad [\text{using (i)}]$$

$$= \int_0^1 (1-x)^2 x^n dx = \int_0^1 (1+x^2-2x)x^n dx = \int_0^1 (x^n + x^{n+2} - 2x^{n+1}) dx$$

$$= \int_0^1 x^n dx + \int_0^1 x^{n+2} dx - 2 \int_0^1 x^{n+1} dx = \left[\frac{x^{n+1}}{n+1} \right]_0^1 + \left[\frac{x^{n+3}}{n+3} \right]_0^1 - 2 \left[\frac{x^{n+2}}{n+2} \right]_0^1$$

$$= \frac{1}{n+1} + \frac{1}{n+3} - \frac{2}{n+2}$$

$$114. \text{ Let } I = \int_0^{\frac{3}{2}} |x \sin \pi x| dx$$

When $0 < x < 1 \Rightarrow 0 < \pi x < \pi \Rightarrow \sin \pi x > 0$

When $1 < x < \frac{3}{2} \Rightarrow \pi < \pi x < \frac{3\pi}{2} \Rightarrow \sin \pi x < 0$

$$\therefore |x \sin \pi x| = \begin{cases} x \sin \pi x, & \text{if } 0 < x < 1 \\ -x \sin \pi x, & \text{if } 1 < x < \frac{3}{2} \end{cases}$$

$$\therefore I = \int_0^1 x \sin \pi x dx - \int_1^{\frac{3}{2}} x \sin \pi x dx$$

$$= \left[-\frac{x \cos \pi x}{\pi} \right]_0^1 + \int_0^1 \frac{\cos \pi x}{\pi} dx + \left[\frac{x \cos \pi x}{\pi} \right]_1^{\frac{3}{2}} - \int_1^{\frac{3}{2}} \frac{\cos \pi x}{\pi} dx$$

$$\dots (i) = \frac{1}{\pi} - 0 + \left[\frac{\sin \pi x}{\pi^2} \right]_0^1 + 0 - \frac{(-1)}{\pi} - \left[\frac{\sin \pi x}{\pi^2} \right]_1^{\frac{3}{2}}$$

$$\dots (ii) = \frac{2}{\pi} + 0 - \frac{(-1)}{\pi^2} + 0 = \frac{2}{\pi} + \frac{1}{\pi^2}$$

[Using (i)]

\dots (iii)

$$115. \text{ Let } I = \int_0^{\frac{\pi}{2}} \frac{x \tan x}{\sec x + \tan x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x \frac{\sin x}{\cos x}}{\frac{1}{\cos x} + \frac{\sin x}{\cos x}} dx = \int_0^{\frac{\pi}{2}} \frac{x \sin x}{1 + \sin x} dx$$

\dots (i)

$$= \int_0^{\frac{\pi}{2}} \frac{(\pi-x) \sin(\pi-x)}{1 + \sin(\pi-x)} dx$$

$$\left[\text{Using the property } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\Rightarrow I = \int_0^{\frac{\pi}{2}} \frac{(\pi-x) \sin x}{1 + \sin x} dx$$

\dots (ii)

Adding (i) and (ii), we get

$$2I = \int_0^{\frac{\pi}{2}} \frac{\pi \sin x}{1 + \sin x} dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x} dx = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{1 + \sin x - 1}{1 + \sin x} dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{1 + \sin x} \right) dx$$

$$= \frac{\pi}{2} \left[x - \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx \right] = \frac{\pi}{2} \left[x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1 - \sin x}{\cos^2 x} dx$$

$$= \frac{\pi}{2} \left[x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \frac{\pi}{2} \left[(\pi - 0) - \int_0^{\frac{\pi}{2}} (\sec^2 x - \tan x \cdot \sec x) dx \right]$$

$$= \frac{\pi}{2} \left[\pi - (\tan x - \sec x) \Big|_0^{\frac{\pi}{2}} \right] = \frac{\pi}{2} [\pi - \{(0 - (-1)) - (0 - 1)\}] = \frac{\pi}{2} [\pi - 2]$$

$$116. \text{ Let } I = \int_1^4 (|x-1| + |x-2| + |x-4|) dx$$

Also, let $f(x) = |x-1| + |x-2| + |x-4|$

We have three critical points $x = 1, 2$ and 4 .

$$f(x) = \begin{cases} (x-1) - (x-2) - (x-4) & \text{if } 1 \leq x < 2 \\ (x-1) + (x-2) - (x-4) & \text{if } 2 \leq x < 4 \end{cases}$$

$$\therefore f(x) = \begin{cases} -x + 5 & \text{if } 1 \leq x < 2 \\ x + 1 & \text{if } 2 \leq x < 4 \end{cases}$$

$$\therefore I = \int_1^4 f(x) dx = \int_1^2 f(x) dx + \int_2^4 f(x) dx$$

$$\begin{aligned}
 &= \int_1^2 (-x+5) dx + \int_2^4 (x+1) dx = \left[-\frac{x^2}{2} + 5x \right]_1^2 + \left[\frac{x^2}{2} + x \right]_2^4 \\
 &= \left(-\frac{4}{2} + 10 \right) - \left(-\frac{1}{2} + 5 \right) + \left(\frac{16}{2} + 4 \right) - \left(\frac{4}{2} + 2 \right) \\
 &= 8 - \frac{9}{2} + 12 - 4 = 16 - \frac{9}{2} = \frac{23}{2}
 \end{aligned}$$

117. Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

$$\begin{aligned}
 \Rightarrow I &= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)} dx \\
 \Rightarrow I &= \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx
 \end{aligned}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left(\frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x} \right) dx \Rightarrow 2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1}{\frac{2 \tan(x/2)}{1 + \tan^2(x/2)} + \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{1 + \tan^2(x/2)}{2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} \cdot \frac{1}{2} dx = dt$

When $x = 0$, then $t = 0$ and when $x = \frac{\pi}{2}$, then $t = 1$

$$\therefore 2I = \int_0^1 \frac{2dt}{2t+1-t^2} = 2 \int_0^1 \frac{1}{(\sqrt{2})^2 - (t-1)^2} dt$$

$$\Rightarrow 2I = 2 \times \frac{1}{2\sqrt{2}} \left[\log \left| \frac{\sqrt{2}+t-1}{\sqrt{2}-t+1} \right| \right]_0^1$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \left\{ \log \left(\frac{\sqrt{2}}{\sqrt{2}} \right) - \log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right\}$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \left\{ 0 - \log \left(\frac{\sqrt{2}-1}{\sqrt{2}+1} \right) \right\} = \frac{1}{\sqrt{2}} \log \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \log \left\{ \frac{(\sqrt{2}+1)^2}{(\sqrt{2}-1)(\sqrt{2}+1)} \right\}$$

$$\Rightarrow 2I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

Key Points

$$\Rightarrow \sin 2A = \frac{2 \tan A}{1 + \tan^2 A} \text{ and } \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

118. Let $I = \int_0^{3/2} |x \cos \pi x| dx$

When $0 < x < \frac{1}{2} \Rightarrow 0 < \pi x < \frac{\pi}{2} \Rightarrow \cos \pi x > 0$

When $\frac{1}{2} < x < \frac{3}{2} \Rightarrow \frac{\pi}{2} < \pi x < \frac{3\pi}{2} \Rightarrow \cos \pi x < 0$

$$\dots(i) \quad \therefore |x \cos \pi x| = \begin{cases} x \cos \pi x, & \text{if } 0 < x < \frac{1}{2} \\ -x \cos \pi x, & \text{if } \frac{1}{2} < x < \frac{3}{2} \end{cases}$$

$$\dots(ii) \quad \therefore I = \int_0^{1/2} x \cos \pi x dx + \int_{1/2}^{3/2} -(x \cos \pi x) dx$$

$$\Rightarrow I = \left[\frac{x}{\pi} \sin \pi x + \frac{\cos \pi x}{\pi^2} \right]_0^{1/2} - \left[\frac{x}{\pi} \sin \pi x + \frac{\cos \pi x}{\pi^2} \right]_{1/2}^{3/2}$$

(Applying integration by parts)

$$= \left[\frac{1}{\pi} \left(\frac{1}{2} - 0 \right) + \frac{1}{\pi^2} (0 - 1) \right] - \left[\frac{1}{\pi} \left(\frac{3}{2} (-1) - \frac{1}{2} (1) \right) + \frac{1}{\pi^2} (0 - 0) \right]$$

$$= \left(\frac{1}{2\pi} - \frac{1}{\pi^2} \right) - \left(\frac{-2}{\pi} \right) = \left(\frac{5\pi - 2}{2\pi^2} \right)$$

119. Let $I = \int_0^{\pi} \frac{x}{1 + \sin \alpha \sin x} dx$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1 + \sin \alpha \sin(\pi - x)} dx \quad \left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi}{1 + \sin \alpha \sin x} dx - \int_0^{\pi} \frac{x}{1 + \sin \alpha \sin x} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi}{1 + \sin \alpha \sin x} dx - I \Rightarrow 2I = \int_0^{\pi} \frac{\pi}{1 + \sin \alpha \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{1 + \sin \alpha \sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{1 + \sin \alpha \left(\frac{2 \tan x/2}{1 + \tan^2 x/2} \right)} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1 + \tan^2 \frac{x}{2}}{\left(1 + \tan^2 \frac{x}{2} + \sin \alpha \times 2 \tan \frac{x}{2} \right)} dx$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{\left(1 + \tan^2 \frac{x}{2} + \sin \alpha \times 2 \tan \frac{x}{2} \right)} dx$$

Put $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} dx = 2dt$

Also, when $x \rightarrow 0$, $t \rightarrow \tan 0 = 0$;

when $x \rightarrow \pi$, $t \rightarrow \tan \frac{\pi}{2} = \infty$

$$\therefore I = \frac{\pi}{2} \int_0^{\infty} \frac{2dt}{t^2 + 2t \sin \alpha + 1}$$

$$\Rightarrow I = \pi \int_0^{\infty} \frac{1}{(t + \sin^2 \alpha)^2 + \cos^2 \alpha} dt$$

$$\Rightarrow I = \frac{\pi}{\cos \alpha} \left[\tan^{-1} \left(\frac{t + \sin \alpha}{\cos \alpha} \right) \right]_0^{\infty}$$

$$\Rightarrow I = \frac{\pi}{\cos \alpha} [\tan^{-1} \infty - \tan^{-1}(\tan \alpha)] \Rightarrow I = \frac{\pi}{\cos \alpha} \left(\frac{\pi}{2} - \alpha \right)$$

120. Let $I = \int_{-\pi}^{\pi} (\cos ax - \sin bx)^2 dx$

$$= \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2 \cos ax \sin bx) dx$$

$$= \int_{-\pi}^{\pi} \cos^2 ax dx + \int_{-\pi}^{\pi} \sin^2 bx dx - 2 \int_{-\pi}^{\pi} \cos ax \sin bx dx$$

$$= 2 \left[\int_0^{\pi} \cos^2 ax dx + \int_0^{\pi} \sin^2 bx dx \right]$$

$$\left[\int_{-a}^a f(x) dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \end{cases} \right]$$

$$= 2 \left[\int_0^{\pi} \left(\frac{1 + \cos 2ax}{2} \right) dx + \int_0^{\pi} \left(\frac{1 - \cos 2bx}{2} \right) dx \right]$$

$$= \int_0^{\pi} (1 + \cos 2ax) dx + \int_0^{\pi} (1 - \cos 2bx) dx$$

$$= 2 \int_0^{\pi} 1 \cdot dx + \int_0^{\pi} \cos 2ax dx - \int_0^{\pi} \cos 2bx dx$$

$$= (2x)_0^{\pi} + \frac{1}{2a} (\sin 2ax)_0^{\pi} - \frac{1}{2b} (\sin 2bx)_0^{\pi}$$

121. Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^{-x} + 1} e^{-x} dx$

$$= \int_{-\pi/2}^{\pi/2} \frac{\cos x (e^{-x} + 1 - 1)}{e^{-x} + 1} dx = \int_{-\pi/2}^{\pi/2} \cos x dx - \int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^{-x} + 1} dx$$

Now, put $x = -z$ in 2nd integral,

$$\therefore dx = -dz$$

Also, when $x = \frac{-\pi}{2}$, then $z = \frac{\pi}{2}$ and when $x = \frac{\pi}{2}$, then $z = \frac{-\pi}{2}$

$$\therefore I = \int_{-\pi/2}^{\pi/2} \cos x dx + \int_{\pi/2}^{-\pi/2} \frac{\cos z}{e^z + 1} dz$$

$$\Rightarrow I = [\sin x]_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} \frac{\cos x}{e^x + 1} dx$$

$$\Rightarrow I = \left[\sin \frac{\pi}{2} + \sin \frac{\pi}{2} \right] - I \Rightarrow 2I = 2 \Rightarrow I = 1$$

122. Let $I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}} = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{\frac{\sin x}{\cos x}}}$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(i)$$

By the property, $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we get

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos \left(\frac{\pi}{2} - x \right)}}{\sqrt{\cos \left(\frac{\pi}{2} - x \right)} + \sqrt{\sin \left(\frac{\pi}{2} - x \right)}} dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right] dx$$

$$= \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

123. Let $I = \int_0^{\pi/4} \log(1 + \tan x) dx$

By using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we get

$$I = \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - x \right) \right] dx$$

$$= \int_0^{\pi/4} \log \left[1 + \frac{1 - \tan x}{1 + \tan x} \right] dx = \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan x} \right) dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx$$

$$\Rightarrow 2I = \log 2 \left[\frac{\pi}{4} - 0 \right] \Rightarrow I = \frac{\pi}{8} \log 2$$

Key Points

$$\Rightarrow \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

124. Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx \quad \dots(i)$

Using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we get

$$I = \int_0^{\pi} \frac{(\pi-x) \tan(\pi-x)}{\sec(\pi-x) \operatorname{cosec}(\pi-x)} dx = \int_0^{\pi} \frac{(\pi-x) \tan x}{\sec x \operatorname{cosec} x} dx \quad \dots(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \left[\frac{x \tan x}{\sec x \operatorname{cosec} x} + \frac{(\pi-x) \tan x}{\sec x \operatorname{cosec} x} \right] dx$$

$$= \pi \int_0^{\pi} \frac{\tan x}{\sec x \operatorname{cosec} x} dx = \pi \int_0^{\pi} \frac{\sin x / \cos x}{\frac{1}{\cos x} \cdot \frac{1}{\sin x}} dx$$

$$= \pi \int_0^{\pi} \sin^2 x dx = \pi \int_0^{\pi} \frac{1 - \cos 2x}{2} dx = \frac{\pi}{2} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi} = \frac{\pi^2}{2} \Rightarrow I = \frac{\pi^2}{4}$$

125. Let $I = \int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x} dx$... (i)

$$I = \int_0^{\pi} \frac{4(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

[By using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$]

$$\Rightarrow I = \int_0^{\pi} \frac{4(\pi - x) \sin x}{1 + \cos^2 x} dx \quad \dots \text{(ii)}$$

Adding (i) and (ii), we get

$$\text{Hence, } 2I = 4\pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

Put $\cos x = t \Rightarrow \sin x dx = -dt$

Also, when $x = 0$, then $t = 1$ and when $x = \pi$, when $t = -1$

$$\therefore I = 2\pi \int_1^{-1} \frac{-dt}{1+t^2} = 2\pi \int_{-1}^1 \frac{dt}{t^2+1} = 2\pi [\tan^{-1} t]_{-1}^1$$

$$= 2\pi [\tan^{-1} 1 - \tan^{-1}(-1)] = 2\pi \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) \right] = \pi^2$$

126. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

$$\therefore I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

Adding (i) and (ii), we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x dx}{\sin^4 x + \cos^4 x}$$

Dividing numerator and denominator by $\cos^4 x$, we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\tan x \sec^2 x dx}{1 + \tan^4 x}$$

Put $\tan^2 x = t \Rightarrow 2 \tan x \sec^2 x dx = dt$

When $x = 0$, then $t = 0$ and when $x = \frac{\pi}{2}$, then $t = \infty$

$$\therefore I = \frac{\pi}{8} \int_0^{\infty} \frac{dt}{1+t^2} = \frac{\pi}{8} [\tan^{-1} t]_0^{\infty} = \frac{\pi}{8} \times \frac{\pi}{2} = \frac{\pi^2}{16}$$

127. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\left[\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right) \right]^{1/2}}{[\sin(\pi/3 + \pi/6 - x)]^{1/2} + [\cos(\pi/3 + \pi/6 - x)]^{1/2}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{[\sin(\pi/2 - x)]^{1/2}}{[\sin(\pi/2 - x)]^{1/2} + [\cos(\pi/2 - x)]^{1/2}} dx$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x} dx}{\sqrt{\sin x} + \sqrt{\cos x}}$$

Adding (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \left(\frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} \right) dx$$

$$\Rightarrow 2I = \int_{\pi/6}^{\pi/3} 1 \cdot dx = [x]_{\pi/6}^{\pi/3} \Rightarrow I = \frac{1}{2} \left[\frac{\pi}{3} - \frac{\pi}{6} \right] \Rightarrow I = \frac{\pi}{12}$$

Concept Applied

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

128. Let $I = \int_0^{\pi} \frac{x dx}{a^2 \cos^2 x + b^2 \sin^2 x}$... (i)

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x) dx}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$$

[Using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$]

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x) dx}{a^2 \cos^2 x + b^2 \sin^2 x} \quad \dots \text{(ii)}$$

Adding (i) and (ii), we get $I = \frac{\pi}{2} \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$

Let $f(x) = \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$

$$\Rightarrow f(\pi - x) = \frac{1}{a^2 \cos^2(\pi - x) + b^2 \sin^2(\pi - x)}$$

... (ii) $\Rightarrow f(\pi - x) = \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} = f(x)$

[By using $\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$, if $f(2a-x) = f(x)$]

$$\therefore I = \frac{\pi}{2} \left(2 \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} \right) \Rightarrow I = \pi \int_0^{\pi/2} \frac{\sec^2 x dx}{a^2 + b^2 \tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

Also, when $x = 0$, then $t = \tan 0 = 0$.

and when $x = \frac{\pi}{2}$, then $t = \tan \frac{\pi}{2} = \infty$

$$\therefore I = \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} \Rightarrow I = \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{\left(\frac{a}{b}\right)^2 + t^2}$$

... (ii) $\Rightarrow I = \frac{\pi}{b^2} \left[\frac{b}{a} \tan^{-1} \left(\frac{bt}{a} \right) \right]_0^{\infty} \Rightarrow I = \frac{\pi}{ab} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{\pi^2}{2ab}$

CBSE Sample Questions

1. Let $I = \int \frac{1}{\cos^2 x (1 - \tan x)^2} dx$

Put $1 - \tan x = t \Rightarrow -\sec^2 x dx = dt$ (1)

$$\therefore I = -\int \frac{dt}{t^2} = -\int t^{-2} dt = \frac{1}{t} + C = \frac{1}{1 - \tan x} + C \quad (1)$$

... (ii) 2. Let $I = \int \frac{\sin 2x}{\sqrt{9 - \cos^4 x}} dx$

Put $\cos^2 x = t$

$$\Rightarrow -2\cos x \sin x dx = dt \Rightarrow \sin 2x dx = -dt$$

$$\therefore I = -\int \frac{dt}{\sqrt{3^2 - t^2}} = -\sin^{-1} \frac{t}{3} + c = -\sin^{-1} \left(\frac{\cos^2 x}{3} \right) + c$$

$$3. \text{ Let } \frac{x+1}{(x^2+1)x} = \frac{Ax+B}{x^2+1} + \frac{C}{x} = \frac{(Ax+B)x+C(x^2+1)}{(x^2+1)x}$$

$$\Rightarrow x+1 = (Ax+B)x + C(x^2+1)$$

By equating the like coefficients, we get

$$B = 1, C = 1, A + C = 0$$

Hence, $A = -1, B = 1$ and $C = 1$

$$\therefore \text{ The given integral} = \int \frac{-x+1}{x^2+1} dx + \int \frac{1}{x} dx$$

$$= \frac{-1}{2} \int \frac{2x-2}{x^2+1} dx + \int \frac{1}{x} dx = \frac{-1}{2} \int \frac{2x}{x^2+1} dx + \int \frac{1}{x^2+1} dx + \int \frac{1}{x} dx$$

$$= \frac{-1}{2} \log(x^2+1) + \tan^{-1} x + \log|x| + c$$

$$4. \text{ Let } I = \int \frac{(x^3+x+1)}{(x^2-1)} dx = \int \left(x + \frac{2x+1}{(x-1)(x+1)} \right) dx$$

Now resolving $\frac{2x+1}{(x-1)(x+1)}$ into partial fractions as

$$\frac{2x+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{x(A+B) + (A-B)}{(x-1)(x+1)}$$

Calculating A and B we get,

$$\frac{2x+1}{(x-1)(x+1)} = \frac{3}{2(x-1)} + \frac{1}{2(x+1)}$$

$$\text{Now, } I = \int \frac{(x^3+x+1)}{(x^2-1)} dx = \int \left(x + \frac{2x+1}{(x-1)(x+1)} \right) dx$$

$$= \int \left(x + \frac{3}{2(x-1)} + \frac{1}{2(x+1)} \right) dx$$

$$= \frac{x^2}{2} + \frac{3}{2} \log|x-1| + \frac{1}{2} \log|x+1| + C$$

$$= \frac{x^2}{2} + \frac{1}{2} \log|(x-1)^3(x+1)| + C$$

Concept Applied

$$\Rightarrow \frac{px+q}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$

$$5. \text{ We have, } \int_0^4 |x-1| dx = \int_0^1 (1-x) dx + \int_1^4 (x-1) dx$$

$$= \left[x - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^4$$

$$= \left(1 - \frac{1}{2} \right) + (8-4) - \left(\frac{1}{2} - 1 \right) = \frac{1}{2} + 4 + \frac{1}{2} = 5$$

$$6. \text{ Let } I = \int \frac{x^2+1}{(x^2+2)(x^2+3)} dx$$

Now, put $x^2 = y$ to make partial fractions.

$$i.e., \frac{x^2+1}{(x^2+2)(x^2+3)} = \frac{y+1}{(y+2)(y+3)} = \frac{A}{y+2} + \frac{B}{y+3}$$

$$\Rightarrow y+1 = A(y+3) + B(y+2) \quad \dots (i) \quad (1/2)$$

(1) Comparing coefficients of y and constant terms on both sides of (i), we get

$$(1) \quad A + B = 1 \text{ and } 3A + 2B = 1$$

Solving, we get $A = -1, B = 2$

$$(1/2) \quad \therefore I = \int \frac{x^2+1}{(x^2+2)(x^2+3)} dx = \int \frac{-1}{x^2+2} dx + 2 \int \frac{1}{x^2+3} dx$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x}{\sqrt{2}} \right) + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3}} \right) + C \quad (1)$$

$$7. \text{ Let } I = \int e^x (1 - \cot x + \operatorname{cosec}^2 x) dx$$

$$(1/2) \quad = \int e^x dx + \int e^x ((-\cot x) + \operatorname{cosec}^2 x) dx$$

$$= e^x + e^x (-\cot x) + C$$

$$= e^x (1 - \cot x) + C \quad (1)$$

$$8. \text{ We have, } \int \frac{\log x}{(1+\log x)^2} dx = \int \frac{\log x + 1 - 1}{(1+\log x)^2} dx$$

$$(1/2) \quad = \int \frac{1}{1+\log x} dx - \int \frac{1}{(1+\log x)^2} dx$$

$$= \frac{1}{1+\log x} \times x - \int \frac{-1}{(1+\log x)^2} \times \frac{1}{x} \times x dx$$

$$- \int \frac{1}{(1+\log x)^2} dx + c = \frac{x}{1+\log x} + c \quad (1\frac{1}{2})$$

9. $\therefore f(x) = x^2 \sin x$ is an odd function.

$$\therefore \int_{-n/2}^{n/2} x^2 \sin x dx = 0 \quad (1)$$

$$10. \text{ Let } I = \int_0^1 x(1-x)^n dx$$

$$(1) \quad \Rightarrow I = \int_0^1 (1-x)[1-(1-x)]^n dx \quad \left(\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right)$$

$$(1/2)$$

$$(1) \quad \Rightarrow I = \int_0^1 (1-x)x^n dx = \int_0^1 (x^n - x^{n+1}) dx \Rightarrow I = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right]_0^1$$

$$(1/2) \quad \Rightarrow I = \left[\left(\frac{1}{n+1} - \frac{1}{n+2} \right) - 0 \right] = \frac{1}{(n+1)(n+2)}$$

11. The given definite integral

$$(1) \quad = \int_{-1}^2 |x(x-1)(x-2)| dx$$

$$(1) \quad = \int_{-1}^0 |x(x-1)(x-2)| dx + \int_0^1 |x(x-1)(x-2)| dx +$$

$$\int_1^2 |x(x-1)(x-2)| dx \quad (1\frac{1}{2})$$

$$(1) \quad = -\int_{-1}^0 (x^3 - 3x^2 + 2x) dx + \int_0^1 (x^3 - 3x^2 + 2x) dx$$

$$- \int_1^2 (x^3 - 3x^2 + 2x) dx \quad (1/2)$$

$$= -\left[\frac{x^4}{4} - x^3 + x^2 \right]_{-1}^0 + \left[\frac{x^4}{4} - x^3 + x^2 \right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2 \right]_1^2$$

$$(1/2) \quad = \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4} \quad (2)$$

Self Assessment

Case Based Objective Questions (4 marks)

1. An Integration is the process of finding the anti-derivative of a function. In this process, we are provided with the derivative of a function and asked to find out the function (i.e., Primitive)
Integration is the inverse process of differentiation. Let $f(x)$ be a function of x . If there is a function $g(x)$, such that $\frac{d}{dx}(g(x)) = f(x)$, then $g(x)$ is called an integral of $f(x)$ w.r.t. x and is denoted by $\int f(x)dx = g(x) + c$, where c is constant of integration.
Also, the given integral $\int f(x)dx$ can be transformed into another form by changing the independent variable x to t by substituting $x = g(t)$
Consider, $I = \int f(x)dx = \int f(g(t))g'(t)dt$
Based on the above information, answer the following questions.

(i) Evaluate: $\int \frac{4x+6}{x^2+3x} dx$

- (a) $3 \log|x+3x^2| + C$ (b) $3 \log|x^2+3x| + C$
(c) $2 \log|x^2+3x| + C$ (d) $\log|4x+6| + C$

(ii) Evaluate: $\int \frac{1+\cos x}{x+\sin x} dx$

- (a) $\log|x+\operatorname{cosec} x|$ (b) $\log|x+\sec x| + C$
(c) $\log|x+\cos x| + C$ (d) $\log|x+\sin x| + C$

(iii) Evaluate: $\int \frac{(x+1)^2}{x(x^2+1)} dx$

- (a) $\log|x| + 2\tan^{-1}x + C$ (b) $2\tan^{-1}x - \log|x| + C$
(c) $\log|x| - 2\tan^{-1}x + C$ (d) None of these

(iv) Evaluate: $\int \tan^2 x dx$

- (a) $\tan x + x + C$ (b) $\tan x - x + C$
(c) $\tan x + x^2 + C$ (d) None of these

(v) Evaluate: $\int \frac{dx}{\sin^2 x \cos^2 x}$

- (a) $-2\cot 2x + C$ (b) $2\cot 2x + C$
(c) $\cot 2x + C$ (d) None of these

Multiple Choice Questions (1 mark)

2. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

- (a) 1 (b) 2 (c) 3 (d) 4

3. $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

- (a) $\frac{\pi^2}{32}$ (b) $\frac{\pi^2}{24}$ (c) $\frac{\pi}{16}$ (d) $\frac{\pi^2}{16}$

OR

Evaluate: $\int_0^1 \left\{ e^x + \sin \frac{\pi x}{4} \right\} dx$

- (a) $1 - \frac{4}{\pi} + \frac{2\sqrt{2}}{\pi}$ (b) $1 + \frac{2}{\pi} - \frac{2\sqrt{2}}{\pi}$
(c) $1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$ (d) None of these

4. $\int_0^{\pi/2} \cos x e^{\sin x} dx$ is equal to _____.

- (a) $e+1$ (b) $e-1$
(c) e^2-1 (d) e^2+1

5. $\int \frac{x^3}{x+1} dx$ is equal to

- (a) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$
(b) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$
(c) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$
(d) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

6. If $\int \frac{dx}{(x+2)(x^2+1)} = a \log|1+x^2|$

$+ b \tan^{-1} x + \frac{1}{5} \log|x+2| + C$, then

- (a) $a = \frac{-1}{10}, b = \frac{-2}{5}$ (b) $a = \frac{1}{10}, b = -\frac{2}{5}$
(c) $a = \frac{-1}{10}, b = \frac{2}{5}$ (d) $a = \frac{1}{10}, b = \frac{2}{5}$

7. $\int e^x \left(\frac{1-x}{1+x^2} \right)^2 dx$ is equal to

- (a) $\frac{e^x}{1+x^2} + C$ (b) $\frac{-e^x}{1+x^2} + C$
(c) $\frac{e^x}{(1+x^2)^2} + C$ (d) $\frac{-e^x}{(1+x^2)^2} + C$

VSA Type Questions (1 mark)

8. Evaluate: $\int \cos^3 x e^{\log \sin x} dx$

9. Evaluate: $\int_1^3 x^2 \log x dx$

10. Evaluate: $\int \sin^{-1} x dx$

OR

Evaluate: $\int \frac{x}{x^4-1} dx$

11. Evaluate: $\int_0^1 \frac{x}{\sqrt{1+x^2}} dx$

12. Evaluate: $\int_0^{\pi} \frac{x}{1+\sin x} dx$

SA I Type Questions

(2 marks)

13. Evaluate: $\int \frac{\sin x}{3+4\cos^2 x} dx$

14. Evaluate: $\int \frac{1+\cos x}{1-\cos x} dx$

15. Evaluate: $\int_0^{\pi/2} \sqrt{1-\sin 2x} dx$

16. Evaluate: $\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx$

OR

If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then find the value of a .

SA II Type Questions

(3 marks)

17. Evaluate: $\int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$

18. Evaluate: $\int_0^1 x \log(1+2x) dx$

19. Evaluate: $\int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$

20. Evaluate: $\int [\sin(\log x) + \cos(\log x)] dx$

OR

Evaluate: $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

21. Evaluate: $\int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}} dx$

Case Based Questions

(4 marks)

22. Let f be a continuous function defined on the closed interval $[a, b]$ and F be an antiderivative of f , then $\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$

This result is very useful as it gives us a method of calculating the definite integral easily. Here, we have no need to write integration constant c because if, we will write $F(x) + c$, instead of $f(x)$, we get

$$\int_a^b f(x) dx = [F(x) + c]_a^b = F(b) + c - F(a) - c = F(b) - F(a)$$

Based on the above information, answer the following questions.

(i) Evaluate: $\int_0^1 xe^x dx$ (ii) Evaluate: $\int_0^{\pi/4} 2 \tan^3 x dx$

LA Type Questions

(4 / 6 marks)

23. Evaluate: $\int \frac{2x-1}{(x-1)(x+2)(x-3)} dx$

24. Evaluate: $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

25. Evaluate: $\int_0^3 (|x| + |x-1| + |x-2|) dx$

OR

Evaluate: $\int \frac{xe^{2x}}{(2x+1)^2} dx$

Detailed SOLUTIONS

1. (i) (c): Let $I = \int \frac{4x+6}{x^2+3x} dx$

$$= 2 \int \frac{2x+3}{x^2+3x} dx$$

Put $x^2 + 3x = t$

$$\Rightarrow (2x+3) dx = dt$$

$$\therefore I = \int 2 \frac{dt}{t} = 2 \log |t| + C$$

$$= 2 \log |x^2 + 3x| + C$$

(ii) (d): Let $I = \int \frac{1+\cos x}{x+\sin x} dx$

Put $x + \sin x = t$

$$\Rightarrow (1 + \cos x) dx = dt$$

$$\therefore I = \int \frac{dt}{t}$$

$$= \log |t| + C$$

$$= \log |x + \sin x| + C$$

(iii) (a): Let $I = \int \frac{(x+1)^2}{x(x^2+1)} dx = \int \frac{x^2+1+2x}{x(x^2+1)} dx$

$$= \int \left(\frac{1}{x} + \frac{2}{x^2+1} \right) dx = \log |x| + 2 \tan^{-1} x + c$$

(iv) (b): $\int \tan^2 x dx = \int (\sec^2 x - 1) dx$
 $= \tan x - x + c$

(v) (a): Let $I = \int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{4}{4 \sin^2 x \cos^2 x} dx$

$$= 4 \int \operatorname{cosec}^2 2x dx = -2 \cot 2x + c$$

2. (a): Let $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x} = \int_{-\pi/4}^{\pi/4} \frac{dx}{2 \cos^2 x}$

$$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x dx = \int_0^{\pi/4} \sec^2 x dx$$

$$\left[\begin{array}{l} \text{Using property for even function } f(x), \\ \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \end{array} \right]$$

$$= [\tan x]_0^{\pi/4} = \left[\tan \frac{\pi}{4} - \tan 0 \right] = 1$$

3. (b): We have, $I = \int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

When $x = 0, t = 0$ and when $x = 1, t = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} \frac{1}{1+x^2} dx = \int_0^{\pi/4} t dt = \left[\frac{t^2}{2} \right]_0^{\pi/4} = \frac{\pi^2}{32}$$

OR

(d): We have, $I = \int_0^1 \left[e^x + \sin \frac{\pi x}{4} \right] dx$

$$= [e^x]_0^1 + \frac{4}{\pi} \left[-\cos \frac{\pi x}{4} \right]_0^1 = e - 1 - \frac{4}{\sqrt{2}\pi} + \frac{4}{\pi}$$

4. (b): Let $I = \int_0^{\pi/2} \cos x e^{\sin x} dx$

Substitute $\sin x = t \Rightarrow \cos x dx = dt$

$x \rightarrow 0 \Rightarrow t \rightarrow 0$

and $x \rightarrow \pi/2 \Rightarrow t \rightarrow 1$

$$\therefore I = \int_0^1 e^t dt = [e^t]_0^1 = e^1 - e^0 = e - 1$$

5. (d): Let $I = \int \frac{x^3}{x+1} dx$

$$= \int \left((x^2 - x + 1) - \frac{1}{(x+1)} \right) dx = \int (x^2 - x + 1) dx - \int \frac{dx}{x+1}$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

6. (c): We have given

$$\int \frac{dx}{(x+2)(x^2+1)} = a \log|1+x^2| + b \tan^{-1} x + \frac{1}{5} \log|x+2| + C$$

Taking, $I = \int \frac{dx}{(x+2)(x^2+1)}$

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

[using method of partial fraction]

$$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x+2)$$

$$\Rightarrow 1 = Ax^2 + A + Bx^2 + 2Bx + Cx + 2C$$

$$\Rightarrow 1 = (A+B)x^2 + (2B+C)x + A+2C$$

$$\Rightarrow A+B=0, A+2C=1, 2B+C=0$$

By solving all three, we get

$$A = \frac{1}{5}, B = -\frac{1}{5} \text{ and } C = \frac{2}{5}$$

$$\therefore \int \frac{dx}{(x+2)(x^2+1)} = \frac{1}{5} \int \frac{1}{x+2} dx + \int \frac{-\frac{1}{5}x + \frac{2}{5}}{x^2+1} dx$$

$$= \frac{1}{5} \int \frac{1}{x+2} dx - \frac{1}{5} \int \frac{x}{1+x^2} dx + \frac{1}{5} \int \frac{2}{1+x^2} dx$$

$$= \frac{1}{5} \log|x+2| - \frac{1}{10} \log|1+x^2| + \frac{2}{5} \tan^{-1} x + C$$

$$\therefore b = \frac{2}{5} \text{ and } a = -\frac{1}{10} \quad [\text{By comparing}]$$

7. (a): We have, $I = \int e^x \left(\frac{1-x}{1+x^2} \right)^2 dx$

$$= \int e^x \left(\frac{1+x^2-2x}{(1+x^2)^2} \right) dx = \int e^x \left(\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2} \right) dx$$

Above integral is of the type $\int e^x (f(x) + f'(x)) dx$

\therefore Solution is $e^x f(x) + C$

$$= e^x \left(\frac{1}{1+x^2} \right) + C$$

8. We have, $I = \int \cos^3 x e^{\log \sin x} dx = \int \cos^3 x \sin x dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt$

$$\Rightarrow I = -\int t^3 dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

9. $\int_1^3 x^2 \log x dx$

$$= \left[(\log x) \left(\frac{x^3}{3} \right) \right]_1^3 - \int_1^3 \frac{31}{x} \cdot \frac{x^3}{3} dx$$

$$= 9 \log 3 - 0 - \frac{1}{3} \left[\frac{x^3}{3} \right]_1^3 = 9 \log 3 - \frac{26}{9}$$

10. Let $I = \int 1 \cdot \sin^{-1} x dx$

$$= (\sin^{-1} x) x - \int \frac{1}{\sqrt{1-x^2}} \cdot x dx + C'$$

Put $1-x^2 = t^2 \Rightarrow -2x dx = 2t dt$

$$= x \sin^{-1} x - \int \frac{(-t dt)}{t} + C = x \sin^{-1} x + \sqrt{1-x^2} + C$$

OR

Let $I = \int \frac{x}{x^4-1} dx$

Substitute $x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{1}{2} dt$

$$\Rightarrow I = \frac{1}{2} \int \frac{dt}{t^2-1} = \frac{1}{2} \cdot \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C$$

$$\left[\text{Using } \int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C \right]$$

$$= \frac{1}{4} [\log|x^2-1| - \log|x^2+1|] + C$$

11. Let $I = \int_0^1 \frac{x}{\sqrt{1+x^2}} dx$

Substitute $1+x^2 = t^2$
 $\Rightarrow 2x dx = 2t dt \Rightarrow x dx = t dt$

$$\Rightarrow I = \int_1^{\sqrt{2}} \frac{t dt}{t} = [t]_1^{\sqrt{2}} = \sqrt{2} - 1$$

12. Let $I = \int_0^{\pi} \frac{x}{1+\sin x} dx$

Using property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we get

$$I = \int_0^{\pi} \frac{\pi-x}{1+\sin(\pi-x)} dx = \int_0^{\pi} \frac{\pi-x}{1+\sin x} dx$$

On adding (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx = \pi \int_0^{\pi} \frac{(1-\sin x)}{(1+\sin x)(1-\sin x)} dx$$

$$= \pi \int_0^{\pi} \frac{(1-\sin x) dx}{\cos^2 x} \quad [\because \cos^2 x = 1 - \sin^2 x]$$

$$= \pi \int_0^{\pi} (\sec^2 x - \tan x \cdot \sec x) dx = \pi [\tan x]_0^{\pi} - \pi [\sec x]_0^{\pi}$$

$$= \pi [\tan \pi - \sec \pi - \tan 0 + \sec 0]$$

$$\Rightarrow 2I = \pi [0 + 1 - 0 + 1] = 2\pi$$

$$\therefore I = \pi$$

13. Let $I = \int \frac{\sin x}{3+4\cos^2 x} dx$

Substitute $\cos x = t \Rightarrow -\sin x dx = dt$

$$\therefore I = -\int \frac{dt}{3+4t^2} = -\frac{1}{4} \int \frac{dt}{\left(\frac{\sqrt{3}}{2}\right)^2 + t^2}$$

$$= -\frac{1}{4} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t}{\sqrt{3}} + C \quad \left[\because \int \frac{1 dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C \right]$$

$$= -\frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{2\cos x}{\sqrt{3}} \right) + C$$

14. Let $I = \int \frac{1+\cos x}{1-\cos x} dx$

$$\Rightarrow I = \int \frac{2\cos^2 \frac{x}{2}}{2\sin^2 \frac{x}{2}} dx = \int \cot^2 \frac{x}{2} dx$$

$$\Rightarrow I = \int \left(\operatorname{cosec}^2 \frac{x}{2} - 1 \right) dx = -2\cot \frac{x}{2} - x + C$$

15. Let $I = \int_0^{\pi/2} \sqrt{1-\sin 2x} dx$

$$= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2} dx + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2} dx$$

$$\left[\begin{array}{l} \because \text{When } 0 < x < \frac{\pi}{4}, \cos x > \sin x \text{ and} \\ \text{when } \frac{\pi}{4} < x < \frac{\pi}{2}, \sin x > \cos x \end{array} \right]$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 + (-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}})$$

$$= 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

... (i)

16. Let $I = \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx$

... (ii)

Now, let $f(x) = x^3 \sin^4 x$, then

$$f(-x) = (-x)^3 (\sin(-x))^4 = -x^3 \sin^4 x = -f(x)$$

So, $f(x)$ is an odd function.

$$\text{Hence, } \int_{-\pi/4}^{\pi/4} x^3 \sin^4 x dx = 0$$

OR

$$\text{We have, } \int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$$

$$\text{Taking } \int_0^a \frac{1}{4\left(\frac{1}{4}+x^2\right)} dx = \frac{1}{4} \int_0^a \frac{1}{\left(x^2+\left(\frac{1}{2}\right)^2\right)} dx$$

$$= \frac{2}{4} [\tan^{-1} 2x]_0^a = \frac{1}{2} \tan^{-1} 2a - 0 = \frac{1}{2} \tan^{-1} 2a$$

$$\text{Since, } \frac{1}{2} \tan^{-1} 2a = \frac{\pi}{8} \Rightarrow \tan^{-1} 2a = \frac{\pi}{4}$$

$$\Rightarrow 2a = \tan \frac{\pi}{4} = 1 \Rightarrow a = 1/2$$

17. Let $I = \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$

$$\text{Let } 2x+1 = A \left(\frac{d}{dx} (x^2+4x+3) \right) + B = A(2x+4) + B$$

Equating the coefficients of x and constant terms, we get

$$2A = 2 \text{ and } 4A + B = 1 \Rightarrow A = 1 \text{ and } B = -3$$

$$\therefore I = \int \frac{2x+1}{\sqrt{x^2+4x+3}} dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+3}} dx - \int \frac{3}{\sqrt{x^2+4x+3}} dx$$

$$= \int \frac{2x+4}{\sqrt{x^2+4x+3}} dx - 3 \int \frac{dx}{\sqrt{(x+2)^2 - (1)^2}}$$

$$= 2\sqrt{x^2+4x+3} - 3 \log |(x+2) + \sqrt{x^2+4x+3}| + C$$

18. Let $I = \int_0^1 x \log(1+2x) dx$

$$= \left[\log(1+2x) \frac{x^2}{2} \right]_0^1 - \int_0^1 \frac{1}{1+2x} \cdot 2 \cdot \frac{x^2}{2} dx$$

[Integration by parts]

$$\begin{aligned}
&= \frac{1}{2} [x^2 \log(1+2x)]_0^1 - \int_0^1 \frac{x^2}{1+2x} dx \\
&= \frac{1}{2} [1 \log 3 - 0] - \left[\int_0^1 \left(\frac{x}{2} - \frac{\frac{x}{2}}{1+2x} \right) dx \right] \\
&= \frac{1}{2} \log 3 - \frac{1}{2} \int_0^1 x dx + \frac{1}{2} \int_0^1 \frac{x}{1+2x} dx \\
&= \frac{1}{2} \log 3 - \frac{1}{2} \left[\frac{x^2}{2} \right]_0^1 + \frac{1}{2} \int_0^1 \frac{1}{2} \frac{(2x+1)-1}{(2x+1)} dx \\
&= \frac{1}{2} \log 3 - \frac{1}{2} \left[\frac{1}{2} - 0 \right] + \frac{1}{4} \int_0^1 dx - \frac{1}{4} \int_0^1 \frac{1}{1+2x} dx \\
&= \frac{1}{2} \log 3 - \frac{1}{4} + \frac{1}{4} [x]_0^1 - \frac{1}{8} [\log|(1+2x)|]_0^1 \\
&= \frac{1}{2} \log 3 - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} [\log 3 - \log 1] \\
&= \frac{1}{2} \log 3 - \frac{1}{8} \log 3 = \frac{3}{8} \log 3
\end{aligned}$$

19. Let $I = \int \frac{\sin(\tan^{-1} x)}{1+x^2} dx$

Put $\tan^{-1} x = t \Rightarrow \frac{1}{1+x^2} dx = dt$

$$\therefore I = \int \frac{\sin(\tan^{-1} x)}{1+x^2} dx = \int \sin t dt = -\cos t + C$$

$$\Rightarrow I = -\cos(\tan^{-1} x) + C$$

20. We have, $I = \int [\sin(\log x) + \cos(\log x)] dx$

Put $\log x = t, x = e^t \Rightarrow dx = e^t dt$

$$\therefore I = \int (\sin t + \cos t) e^t dt$$

Consider, $f(t) = \sin t$

$$\Rightarrow f'(t) = \cos t$$

$$\therefore \text{Integrand is in the form } e^t(f(t) + f'(t))$$

$$\therefore I = \int e^t (\sin t + \cos t) dt = e^t \sin t + C = x \sin(\log x) + C$$

OR

$$\text{Let } I = \int \frac{\cos x - \cos 2x}{1 - \cos x} dx$$

$$= \int \frac{2 \sin \frac{3x}{2} \cdot \sin \frac{x}{2}}{1 - 1 + 2 \sin^2 \frac{x}{2}} dx \quad \left[\begin{array}{l} \text{Using } \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2} \\ \text{and } \cos A = 1 - 2 \sin^2 \frac{A}{2} \end{array} \right]$$

$$= \int \frac{\sin \frac{3x}{2}}{\sin \frac{x}{2}} dx$$

$$= \int \frac{3 \sin \frac{x}{2} - 4 \sin^3 \frac{x}{2}}{\sin \frac{x}{2}} dx \quad [\because \sin 3A = 3 \sin A - 4 \sin^3 A]$$

$$\begin{aligned}
&= 3 \int dx - 4 \int \sin^2 \frac{x}{2} dx \\
&= 3 \int dx - 4 \int \frac{1 - \cos x}{2} dx \\
&= 3 \int dx - 2 \int dx + 2 \int \cos x dx \\
&= \int dx + 2 \int \cos x dx = x + 2 \sin x + C
\end{aligned}$$

21. We have, $I = \int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}} dx$

Put $\cos^2 2x = t \Rightarrow -4 \sin 2x \cos 2x dx = dt$

$$\therefore I = \frac{-1}{4} \int \frac{dt}{\sqrt{9-t^2}} = \frac{-1}{4} \sin^{-1} \frac{t}{3} + C$$

$$= \frac{-1}{4} \sin^{-1} \left(\frac{\cos^2 2x}{3} \right) + C$$

22. (i) Here, $\int x e^x dx = x \int e^x dx - \int \left(\frac{d}{dx}(x) \cdot \int e^x dx \right) dx$

$$= x e^x - \int 1 \cdot e^x dx$$

$$= x e^x - e^x = e^x (x - 1)$$

$$= F(x)$$

$$\text{Now, } \int_0^1 x e^x dx = F(1) - F(0)$$

$$= e(1-1) - e^0(0-1) = 0 + 1 = 1$$

(ii) We have, $\int 2 \tan^3 x dx$

$$= \int 2 \tan x \tan^2 x dx$$

$$= \int 2 \tan x (\sec^2 x - 1) dx$$

$$= 2 \int \tan x \sec^2 x dx - 2 \int \tan x dx$$

$$= 2 \left[\frac{\tan^2 x}{2} \right] - 2[-\log|\cos x|] = \tan^2 x + 2 \log|\cos x|$$

$$\text{Now, } \int_0^{\pi/4} 2 \tan^3 x = F(\pi/4) - F(0)$$

$$= \left(\tan^2 \frac{\pi}{4} + 2 \log \left| \cos \frac{\pi}{4} \right| \right) - (\tan^2 0 + 2 \log|\cos 0|)$$

$$= \left(1 + 2 \log \frac{1}{\sqrt{2}} \right) - (0 + 2 \log 1) = 1 + 2 \left(\frac{-1}{2} \log 2 \right) - 0$$

$$= 1 - \log 2$$

23. Let $I = \int \frac{(2x-1)}{(x-1)(x+2)(x-3)} dx$

By using partial fraction, we get

$$\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$$

$$\Rightarrow 2x-1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$$

Substitute $x = 1$, we get

$$2-1 = A(1+2)(1-3)$$

$$\Rightarrow 1 = -6A \Rightarrow A = -\frac{1}{6}$$

Substitute $x = 3$, we get

$$6 - 1 = C(3 - 1)(3 + 2)$$

$$\Rightarrow 5 = 10C \Rightarrow C = \frac{1}{2}$$

Now, substitute $x = -2$, we get

$$-4 - 1 = B(-2 - 1)(-2 - 3)$$

$$\Rightarrow -5 = 15B \Rightarrow B = -\frac{1}{3}$$

$$\therefore I = -\frac{1}{6} \int \frac{1}{x-1} dx - \frac{1}{3} \int \frac{1}{x+2} dx + \frac{1}{2} \int \frac{1}{x-3} dx$$

$$= -\frac{1}{6} \log|x-1| - \frac{1}{3} \log|x+2| + \frac{1}{2} \log|x-3| + C$$

$$= -\log|(x-1)^{1/6} - \log|(x+2)^{1/3} + \log|\sqrt{(x-3)}| + C$$

$$= \log|\sqrt{x-3}| - \log|(x-1)^{1/6} (x+2)^{1/3}| + C$$

$$= \log \left| \frac{\sqrt{x-3}}{(x-1)^{1/6} (x+2)^{1/3}} \right| + C$$

24. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)}{\sin^2 x \cdot \cos^2 x} dx$$

$$[\because a^3 + b^3 = (a+b)(a^2 - ab + b^2) \text{ and } \sin^2 x + \cos^2 x = 1]$$

$$= \int \frac{\sin^4 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x \cdot \cos^2 x} dx - \int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cdot \cos^2 x} dx$$

$$= \int \tan^2 x dx + \int \cot^2 x dx - \int 1 dx$$

$$= \int (\sec^2 x - 1) dx + \int (\operatorname{cosec}^2 x - 1) dx - \int 1 dx$$

$$= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx - 3 \int dx$$

$$\Rightarrow I = \tan x - \cot x - 3x + C$$

25. Let $I = \int_0^3 (|x| + |x-1| + |x-2|) dx$

$$f(x) = |x| + |x-1| + |x-2|$$

When $0 \leq x < 1$, then

$$f(x) = x - (x-1) - (x-2) = x - x + 1 - x + 2 = -x + 3$$

When $1 \leq x < 2$, then

$$f(x) = x + (x-1) - (x-2) = x + x - 1 - x + 2 = x + 1$$

When $2 \leq x < 3$, then

$$f(x) = x + (x-1) + (x-2) = 3x - 3$$

$$\therefore f(x) = \begin{cases} -x+3, & 0 \leq x < 1 \\ x+1, & 1 \leq x < 2 \\ 3x-3, & 2 \leq x < 3 \end{cases}$$

$$\therefore I = \int_0^3 (|x| + |x-1| + |x-2|) dx$$

$$\Rightarrow I = \int_0^1 (-x+3) dx + \int_1^2 (x+1) dx + \int_2^3 (3x-3) dx$$

$$= \left[\frac{-x^2}{2} + 3x \right]_0^1 + \left[\frac{x^2}{2} + x \right]_1^2 + \left[\frac{3x^2}{2} - 3x \right]_2^3$$

$$= \left(\frac{-1}{2} + 3 \right) + \left[\left(\frac{4}{2} + 2 \right) - \left(\frac{1}{2} + 1 \right) \right] + \left[\left(\frac{27}{2} - 9 \right) - \left(\frac{12}{2} - 6 \right) \right]$$

$$= \frac{5}{2} + \left[4 - \frac{3}{2} \right] + \left[\frac{9}{2} - 0 \right] = \frac{5}{2} + 4 - \frac{3}{2} + \frac{9}{2} = \frac{19}{2}$$

OR

$$\text{Let } I = \int \frac{x e^{2x}}{(2x+1)^2} dx$$

$$\text{Put } 2x = t \Rightarrow dx = \frac{dt}{2}$$

$$\therefore I = \frac{1}{4} \int \frac{t e^t}{(t+1)^2} dt = \frac{1}{4} \int e^t \left[\frac{t+1}{(t+1)^2} - \frac{1}{(t+1)^2} \right] dt$$

$$= \frac{1}{4} \int e^t \left(\frac{1}{t+1} - \frac{1}{(t+1)^2} \right) dt$$

$$\text{If } f(t) = \frac{1}{t+1} \Rightarrow f'(t) = \frac{-1}{(t+1)^2}$$

$$\text{So, } I = \frac{1}{4} \frac{e^{2x}}{(2x+1)} + C$$