

Application of Derivatives

(2025)

Q.1 The values of λ so that $f(x) = \sin x - \cos x - \lambda x + C$ decreases for all real values of x are :

(1 Mark) (CBSE 2025 - 65/4/1)

A. $1 < \lambda < \sqrt{2}$

B. $\lambda \geq \sqrt{2}$

C. $\lambda \geq 1$

D. $\lambda < 1$

Q.2 If $f(x) = 2x + \cos x$, then $f(x)$:

(1 Mark) (CBSE 2025 - 65/4/1)

A. is an increasing function

B. has a minima at $x = \pi$

C. is a decreasing function

D. has a maxima at $x = \pi$

Q.3 Let $f(x) = |x|$, $x \in \mathbb{R}$. Then, which of the following statements is incorrect?

(1 Mark) (CBSE 2025 - 65/6/1)

A. f has a minimum value at $x = 0$.

B. f is differentiable at $x = 0$.

C. f has no maximum value in \mathbb{R} .

D. f is continuous at $x = 0$.

Q.4 The function $f(x) = x^2 - 4x + 6$ is increasing in the interval

(1 Mark) (CBSE 2025 - 65/2/1)

- A. $(-\infty, 2]$
- B. $[1, 2]$
- C. $(0, 2)$
- D. $[2, \infty)$

Q.5 A cylindrical tank of radius 10 cm is being filled with sugar at the rate of $100\pi \text{ cm}^3/\text{s}$. The rate, at which the height of the sugar inside the tank is increasing, is :

(1 Mark) (CBSE 2025 - 65/2/1)

- A. 1 cm/s
- B. 1.1 cm/s
- C. 0.5 cm/s
- D. 0.1 cm/s

Q.6 A spherical ball has a variable diameter $\frac{5}{2}(3x + 1)$. The rate of change of its volume w.r.t. x , when $x = 1$, is :

(1 Mark) (CBSE 2025 - 65/7/1)

- A. 300π
- B. 375π
- C. 125π
- D. 225π

Q.7 If $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $f(x) = 2x - \sin x$, then f is :

(1 Mark) (CBSE 2025 - 65/7/1)

- A. a decreasing function
- B. maximum at $x = 0$
- C. an increasing function
- D. maximum at $x = \pi/2$

Q.8 The slope of the curve $y = -x^3 + 3x^2 + 8x - 20$ is maximum at :

(1 Mark) (CBSE 2025 - 65/5/1)

A. $(-10, 1)$

B. $(1, -10)$

C. $(1, 10)$

D. $(10, 1)$

Q.9 The absolute maximum value of function $f(x) = x^3 - 3x + 2$ in $[0,2]$ is :

(1 Mark) (CBSE 2025 - 65/1/1)

A. 2

B. 4

C. 5

D. 0

Q.10 Find the least value of 'a' so that $f(x) = 2x^2 - ax + 3$ is an increasing function on $[2,4]$.

(2 Mark) (CBSE 2025 - 65/4/1)

Q.11

If $f(x) = x + \frac{1}{x}$, $x \geq 1$, show that f is an increasing function.

(2 Mark) (CBSE 2025 - 65/4/1)

Q.12 For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units /s, then how fast is the slope of the curve changing when $x=2$?

(2 Mark) (CBSE 2025 - 65/4/1)

Q.13 Determine the values of x for which $f(x) = \frac{x-4}{x+1}$, $x \in \mathbb{R} - 1$ is an increasing or a decreasing function.

(2 Mark) (CBSE 2025 - 65/6/1)

Q.14 Find the values of 'a' for which $f(x) = \sin x - ax + b$ is increasing on \mathbb{R} .

(2 Mark) (CBSE 2025 - 65/6/1)

Q.15 Surface area of a balloon (spherical), when air is blown into it, increases at a rate of $5 \text{ mm}^2/\text{s}$. When the radius of the balloon is 8 m, find the rate at which the volume of the balloon is increasing.

(2 Mark) (CBSE 2025 - 65/5/1)

Q.16 Find the intervals in which function $f(x) = 5x^{\frac{3}{2}} - 3x^{\frac{5}{2}}$ is (i) increasing (ii) decreasing.

(2 Mark) (CBSE 2025 - 65/1/1)

Q.17 Amongst all pairs of positive integers with product as 289, find which of the two numbers add up to the least.

(3 Mark) (CBSE 2025 - 65/7/1)

Q.18 Find the value of 'a' for which $f(x) = \sqrt{3}\sin x - \cos x - 2ax + 6$ is decreasing in \mathbb{R} .

(3 Mark) (CBSE 2025 - 65/5/1)

Q.19 The side of an equilateral triangle is increasing at the rate of 3 cm/s. At what rate its area increasing when the side of the triangle is 15 cm ?

(3 Mark) (CBSE 2025 - 65/1/1)

Q.20



A small town is analyzing the pattern of a new street light installation. The lights are set up in such a way that the intensity of light at any point x metres from the start of the street can be modelled

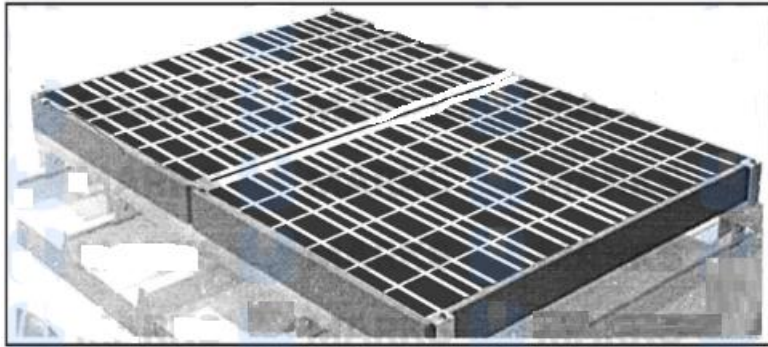
(4 Mark) (CBSE 2025 - 65/2/1)

by $f(x) = e^x \sin x$, where x is in metres.

Based on the above, answer the following :

- (i) Find the intervals on which the $f(x)$ is increasing or decreasing, $x \in [0, \pi]$.
- (ii) Verify, whether each critical point when $x \in [0, \pi]$ is a point of local maximum or local minimum or a point of inflexion.

Q.21



A technical company is designing a rectangular solar panel installation on a roof using 300 metres of boundary material. The design includes a partition running parallel to one of the sides dividing the area (roof) into two sections.

(4 Mark) (CBSE 2025 - 65/1/1)

Let the length of the side perpendicular to the partition be x metres and with parallel to the partition be y metres.

Based on this information, answer the following questions :

- (i) Write the equation for the total boundary material used in the boundary and parallel to the partition in terms of x and y .
- (ii) Write the area of the solar panel as a function of x .
- (iii) (a) Find the critical points of the area function. Use second derivative test to determine critical points at the maximum area. Also, find the maximum

area.

(iii) (b) Using first derivative test, calculate the maximum area the company can enclose with the 300 metres of boundary material, considering the parallel partition.

Q.22 A carpenter needs to make a wooden cuboidal box, closed from all sides, which has a square base and fixed volume. Since he is short of the paint required to paint the box on completion, he wants the surface area to be minimum.

On the basis of the above information, answer the following questions :

(i) Taking length = breadth = x m and height = y m, express the surface area (S) of the box in terms of x and its volume (V), which is constant.

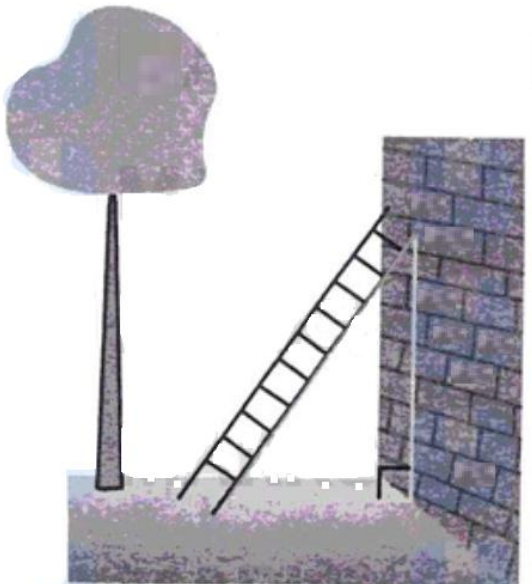
(4 Mark) (CBSE 2025 - 65/5/1)

(ii) Find $\frac{dS}{dx}$.

(iii) (a) Find a relation between x and y such that the surface area (S) is minimum.

(iii) (b) If surface area (S) is constant, the volume (V) = $\frac{1}{4}(Sx - 2x^3)$, x being the edge of base. Show that volume (v) is maximum for $x = \sqrt{\frac{S}{6}}$.

Q.23



(4 Mark) (CBSE 2025 - 65/6/1)

A ladder of fixed length 'h' is to be placed along the wall such that it is free to move along the height of the wall.

Based upon the above information, answer the following questions :

(i) Express the distance (y) between the wall and foot of the ladder in terms of 'h' and height (x) on the wall at a certain instant. Also, write an expression in terms of h and x for the area (A) of the right triangle, as seen from the side by an observer.

(ii) Find the derivative of the area (A) with respect to the height on the wall (x), and find its critical point.

(iii) (a) Show that the area (A) of the right triangle is maximum at the critical point.

(iii) (b) If the foot of the ladder whose length is 5 m, is being pulled towards the wall such that the rate of decrease of distance (y) is 2 m/s, then at what rate is the height on the wall (x) increasing, when the foot of the ladder is 3 m away from the wall?

Q.24 The relation between the height of the plant (y cm) with respect to exposure to sunlight is governed by the equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed to sunlight.

(4 Mark) (CBSE 2025 - 65/4/1)

(i) Find the rate of growth of the plant with respect to sunlight.

(ii) In how many days will the plant attain its maximum height? What is the maximum height?

Q.25 Find the absolute maximum and absolute minimum of function $f(x) = 2x^3 - 15x^2 + 36x + 1$ on [1,5].

(5 Mark) (CBSE 2025 - 65/5/1)

Answer

Q.1 B

Q.2 A

Q.3 B

Q.4 D

Q.5 A

Q.6 B

Q.7 C

Q.8 B

Q.9 B

Q.10

$$f(x) = 2x^2 - ax + 3 \Rightarrow f'(x) = 4x - a$$

$$\text{Now } 2 \leq x \leq 4 \Rightarrow 8 - a \leq 4x - a \leq 16 - a$$

For f to be an increasing function, $f'(x) \geq 0$

$$\Rightarrow 8 - a \geq 0 \Rightarrow a \leq 8$$

\therefore Least value of a does not exist.

Q.11

$$f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

Now $\frac{x^2-1}{x^2} \geq 0$ for all $x \geq 1 \Rightarrow f'(x) \geq 0 \Rightarrow f$ is an increasing function.

Q.12

$$y = 5x - 2x^3$$

$$\text{Given } \frac{dx}{dt} = 2 \text{ units /s}$$

$$\text{slope of the curve} = \frac{dy}{dx} = 5 - 6x^2 = m$$

$$\frac{dm}{dt} = -12x \frac{dx}{dt} = -12x(2) = -24x$$

$$\text{at } x = 2, \frac{dm}{dt} = -24(2) = -48$$

Hence, slope of curve is decreasing at the rate of 48

Q.13

$$f'(x) = \frac{x+1-x+4}{(x+1)^2} = \frac{5}{(x+1)^2} > 0$$

Hence f is increasing in its domain.

Q.14

$$f'(x) = \cos x - a$$

For $f(x)$ to be increasing, $f'(x) \geq 0$

$$\text{i.e., } \cos x \geq a$$

Since, $-1 \leq \cos x \leq 1$

$$\Rightarrow a \leq -1$$

Hence, $a \in (-\infty, -1]$. (Also, accept $a \in (-\infty, -1)$)

Q.15

$$\frac{dS}{dt} = 5 \text{ mm}^2/\text{s}, \quad \left(\frac{dV}{dt} \right)_{r=8} = ?$$

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{8\pi r}$$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = \frac{5}{2}r$$

$$\Rightarrow \left(\frac{dV}{dt} \right)_{r=8} = 20 \text{ mm}^3/\text{s}$$

Q.16

$$f(x) = 5x^{3/2} - 3x^{5/2} \Rightarrow f'(x) = \frac{15}{2}\sqrt{x}(1-x)$$

For increasing/decreasing, put $f'(x) = 0$

$$\Rightarrow x = 0, 1$$

- (i) When $x \in [0, 1]$, $f'(x) \geq 0$. So, f is increasing when $x \in [0, 1]$
 (The intervals $(0, 1)$, $[0, 1)$ or $(0, 1]$ can also be considered.)
- (ii) When $x \in [1, \infty)$, $f'(x) \leq 0$. So, f is decreasing when $x \in [1, \infty)$
 (The interval $(1, \infty)$ can also be considered.)

Q.17

Let numbers be ' x ' and ' y ' such that $xy = 289 \Rightarrow y = \frac{289}{x}$, ' S ' be their sum, then

$$S = x + y = x + \frac{289}{x}$$

$$\frac{dS}{dx} = 1 - \frac{289}{x^2}, \frac{dS}{dx} = 0 \Rightarrow x = 17, \text{ a positive integer}$$

$$\left. \frac{d^2 S}{dx^2} \right|_{x=17} = 289 \left(\frac{2}{x^3} \right) \Big|_{x=17} > 0, \therefore S \text{ is minimum when } x = 17, y = 17$$

Q.18 Since $f(x)$ is a decreasing function

$$\Rightarrow f'(x) \leq 0$$

$$\Rightarrow \sqrt{3} \cdot \cos x + \sin x - 2a \leq 0$$

$$\Rightarrow 2 \left(\frac{\sqrt{3}}{2} \cdot \cos x + \frac{1}{2} \sin x \right) - 2a \leq 0$$

$$\Rightarrow \cos \left(x - \frac{\pi}{6} \right) \leq a$$

Since,

$$-1 \leq \cos \left(x - \frac{\pi}{6} \right) \leq 1 \Rightarrow a \geq 1$$

i.e. a

$$\in [1, \infty)$$

or

$$(1, \infty)$$

Q.19

Let ' a '

' be the side of the triangle, so

' be the side of the triangle, so

$$\frac{da}{dt} = 3 \text{ cm/s}$$

Now area of an equilateral triangle,

$$A = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}a}{2} \times \frac{da}{dt}$$

$$\therefore \left. \frac{dA}{dt} \right]_{a=15 \text{ cm}} = \frac{\sqrt{3} \times 15}{2} \times 3 = \frac{45\sqrt{3}}{2} \text{ cm}^2/\text{s}$$

Q.20

$$f'(x) = e^x(\cos x + \sin x)$$

For critical points,

$$f'(x) = 0$$

$$\Rightarrow \cos x + \sin x = 0$$

$$\Rightarrow \cos x = -\sin x$$

For x to be a critical point

$$x \in (0, \pi)$$

, hence,

$$x = \frac{3\pi}{4}$$

For all

$$x \in \left[0, \frac{3\pi}{4}\right], f'(x) \geq 0$$

Hence,

f

is increasing in

$$\left[0, \frac{3\pi}{4}\right]$$

Note: If a student concludes the answer in any of the following intervals, full marks may be awarded:

$$\left(0, \frac{3\pi}{4}\right)$$

or

$$\left[0, \frac{3\pi}{4}\right)$$

or

$$\left(0, \frac{3\pi}{4}\right]$$

For all

$$x \in \left[\frac{3\pi}{4}, \pi\right], f'(x) \leq 0$$

Hence, f is decreasing in

$$\left[\frac{3\pi}{4}, \pi\right]$$

Note: If a student concludes the answer in any of the following intervals, full marks may be awarded:

$$\left(\frac{3\pi}{4}, \pi\right) \text{ or } \left(\frac{3\pi}{4}, \pi\right] \text{ or } \left[\frac{3\pi}{4}, \pi\right)$$

$$x = \frac{3\pi}{4}$$

is a critical point

$$\begin{aligned} f''(x) &= e^x(\cos x - \sin x) + e^x(\cos x + \sin x) \\ &= 2e^x \cos x \end{aligned}$$

$$f''\left(\frac{3\pi}{4}\right) = -ve$$

Hence,

$$\frac{3\pi}{4}$$

is a point of local maximum.

Q.21

(i) $2x + 3y = 300$

(ii) $A = xy = \frac{x}{3}(300 - 2x)$

(iii) (a) $A = \frac{x}{3}(300 - 2x) = \frac{1}{3}(300x - 2x^2)$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{3}(300 - 4x)$$

For critical points, put

$$\frac{dA}{dx} = 0 \Rightarrow x = 75$$

Also,

$$\frac{d^2A}{dx^2} = -\frac{4}{3} < 0$$

. So, A is maximum at

$$x = 75$$

Also, maximum area is

$$A = \frac{75}{3}(300 - 150) = 3750 \text{ m}^2$$

(iii)(b)

(iii) (b) $A = \frac{X}{3}(300 - 2x) = \frac{1}{3}(300x - 2x^2)$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{3}(300 - 4x)$$

For critical points, put

$$\frac{dA}{dx} = 0 \Rightarrow x = 75$$

As

$$\frac{dA}{dx}$$

changes its sign from positive to negative as x

passes through

$$x = 75$$

from left to right, which means

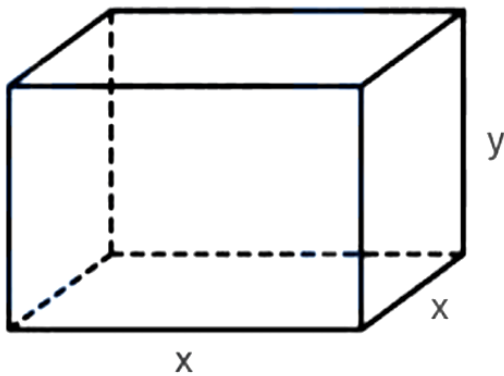
$$x = 75$$

is the point of maximum.

Also, maximum area is

$$A = \frac{75}{3}(300 - 150) = 3750 \text{ m}^2$$

Q.22



$$(i) \quad V = x^2y \Rightarrow y = \frac{V}{x^2}$$

$$\text{Hence, } S = 2x^2 + 4xy = 2x^2 + \frac{4V}{x}$$

$$(ii) \quad \frac{dS}{dx} = 4 \left(x - \frac{V}{x^2} \right)$$

$$(iii) (a) \quad \frac{dS}{dx} = 0 \Rightarrow V = x^3 \Rightarrow x^2y = x^3 \Rightarrow y = x$$

$$\frac{d^2S}{dx^2} = 4 \left(1 + \frac{2V}{x^3} \right) > 0 \Rightarrow S \text{ is minimum if } y = x.$$

(iii) (b)

$$V = \frac{1}{4}(Sx - 2x^3) \Rightarrow \frac{dV}{dx} = \frac{1}{4}(S - 6x^2)$$

$$\text{Put } \frac{dV}{dx} = 0 \Rightarrow x = \sqrt{\frac{S}{6}}$$

$$\left(\frac{d^2V}{dx^2} \right)_{x=\sqrt{\frac{S}{6}}} = -3\sqrt{\frac{S}{6}} < 0 \Rightarrow \text{Volume is maximum for } x = \sqrt{\frac{S}{6}}.$$

Q.23

(i)

$$y^2 = h^2 - x^2$$

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$$

(ii)

$$\frac{dA}{dx} = \frac{1}{2}\sqrt{h^2 - x^2} + \frac{1}{2}x \frac{-x}{\sqrt{h^2 - x^2}} \frac{dA}{dx} = 0$$

gives

$$x = \frac{h}{\sqrt{2}}$$

(iii) (a) A''

$$= \frac{1}{2} \frac{-4x \cdot \sqrt{h^2 - x^2} - (h^2 - 2x^2) \frac{-x}{\sqrt{h^2 - x^2}}}{h^2 - x^2}$$

is

< 0

at

$$x = \frac{h}{\sqrt{2}}$$

Hence A is maximum at critical point

(iii)(b) $y^2 = 25 - x^2$ hence $y = 3$ gives $x = 4$

$$2y \frac{dy}{dt} = -2x \frac{dx}{dt}$$

$$\frac{dx}{dt} = 1.5 \text{ m/s}$$

Q.24

$$(i) y = 4x - \frac{1}{2}x^2 \Rightarrow \frac{dy}{dx} = (4 - x) \text{ cm/day}$$

(ii) For maximum height, $\frac{dy}{dx} = 0 \Rightarrow x = 4$ days as $\frac{d^2y}{dx^2} < 0$, number of days = 4Now, Maximum height = $y(4) = 16 - \frac{1}{2}(16) = 8 \text{ cm}$

Q.25

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$\Rightarrow f'(x) = 6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$$

$$f'(x) = 0 \Rightarrow x = 2, 3 \in [1, 5]$$

Now

$$f(1) = 24, f(2) = 29, f(3) = 28, f(5) = 56$$

Hence, the absolute maximum value is 56 and the absolute minimum value is 24.

(2024)

Q.1 The function $f(x) = x^3 - 3x^2 + 12x - 18$:

(1 Mark) (CBSE 2025 - 65/2/1)

- A. strictly decreasing on $(-\infty, 0)$
- B. strictly decreasing on \mathbb{R}
- C. neither strictly increasing nor strictly decreasing on \mathbb{R}
- D. strictly increasing on \mathbb{R}

Q.2 The function $f(x) = kx - \sin x$ is strictly increasing for

(1 Mark) (CBSE 2025 - 65/4/1)

- A. $k > 1$
- B. $k < 1$
- C. $k > -1$
- D. $k < -1$

Q.3 Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) .

Then, this function $f(x)$ is strictly increasing in (a, b) if

(1 Mark) (CBSE 2025 - 65/1/1)

- A. $f'(x) > 0, \forall x \in (a, b)$
- B. $f'(x) = 0, \forall x \in (a, b)$
- C. $f(x) > 0, \forall x \in (a, b)$
- D. $f'(x) < 0, \forall x \in (a, b)$

Q.4 If the sides of a square are decreasing at the rate of 1.5 cm/s, the rate of decrease of its perimeter is :

(1 Mark) (CBSE 2025 - 65/3/1)

- A. 3 cm/s

- B. 6 cm/s
- C. 1.5 cm/s
- D. 2.25 cm/s

Q.5

The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minima at x equal to :

(1 Mark) (CBSE 2025 - 65/5/1)

- A. -2
- B. 0
- C. 2
- D. 1

Q.6 Given a curve $y = 7x - x^3$ and x increases at the rate of 2 units per second. The rate at which the slope of the curve is changing, when $x = 5$ is:

(1 Mark) (CBSE 2025 - 65/5/1)

- A. 60 units/sec
- B. -60 units/sec
- C. -70 units/sec
- D. -140 units/sec

Q.7 If M and m denote the local maximum and local minimum values of the function $f(x) = x + \frac{1}{x}$ ($x \neq 0$) respectively, find the value of $(M - m)$.

(2 Mark) (CBSE 2024 - 65/2/1)

Q.8 Show that $f(x) = e^x - e^{-x} + x - \tan^{-1} x$ is strictly increasing in its domain.

(2 Mark) (CBSE 2024 - 65/2/1)

Q.9 Find the interval in which the function $f(x) = x^4 - 4x^3 + 10$ is strictly decreasing.

(2 Mark) (CBSE 2024 - 65/4/1)

Q.10 The volume of a cube is increasing at the rate of $6 \text{ cm}^3/\text{s}$. How fast is the surface area of cube increasing, when the length of an edge is 8 cm ?

(2 Mark) (CBSE 2024 - 65/4/1)

Q.11 Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

(2 Mark) (CBSE 2024 - 65/1/1)

Q.12 The area of the circle is increasing at a uniform rate of $2 \text{ cm}^2/\text{sec}$. How fast is the circumference of the circle increasing when the radius $r = 5 \text{ cm}$?

(2 Mark) (CBSE 2024 - 65/5/1)

Q.13 Find the intervals in which the function $f(x) = \frac{\log x}{x}$ is strictly increasing or strictly decreasing.

(4 Mark) (CBSE 2024 - 65/3/1)

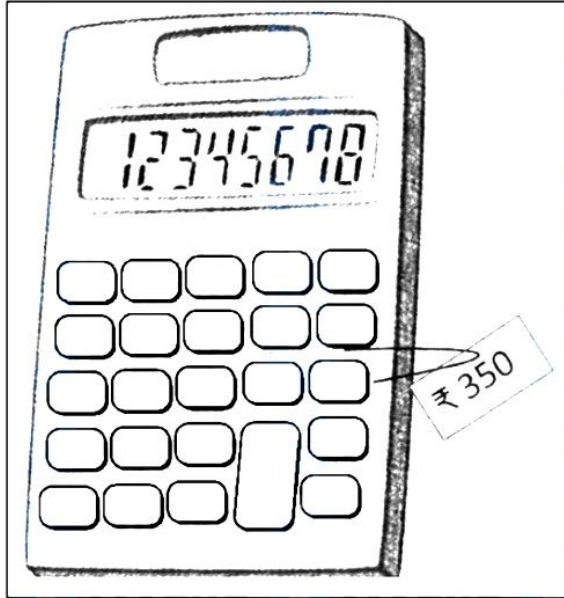
Q.14 Find the absolute maximum and absolute minimum values of the function f given by $f(x) = \frac{x}{2} + \frac{2}{x}$, on the interval $[1, 2]$.

(3 Mark) (CBSE 2024 - 65/3/1)

Q.15 A store has been selling calculators at ₹ 350 each. A market survey indicates that a reduction in price (p) of calculator increases the number of units (x) sold. The relation between the price and quantity sold is given by the

demand function $p = 450 - \frac{1}{2}x$.

(4 Mark) (CBSE 2024 - 65/4/1)

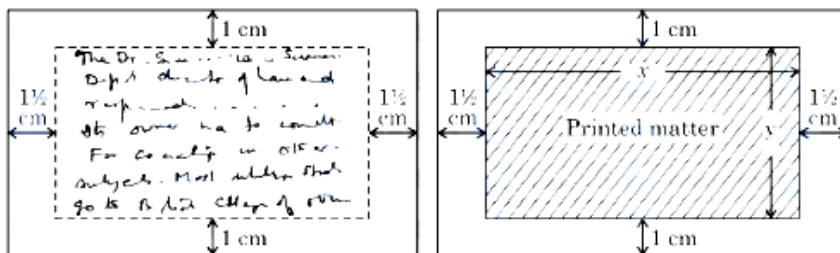


Based on the above information, answer the following questions :

- (i) Determine the number of units (x) that should be sold to maximise the revenue $R(x) = xp(x)$. Also, verify the result.
- (ii) What rebate in price of calculator should the store give to maximise the revenue?

Q.16 A rectangular visiting card is to contain 24 sq.cm. of printed matter. The margins at the top and bottom of the card are to be 1 cm and the margins on the left and right are to be $1\frac{1}{2}$ cm as shown below :

(4 Mark) (CBSE 2024 - 65/5/1)

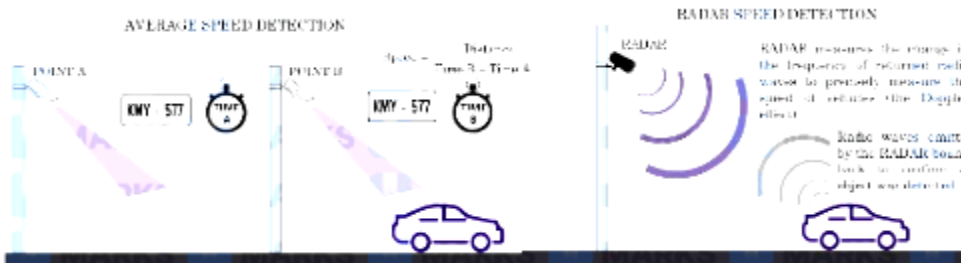


On the basis of the above information, answer the following questions :

- (i) Write the expression for the area of the visiting card in terms of x .
- (ii) Obtain the dimensions of the card of minimum area.

Q.17 The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.

(4 Mark) (CBSE 2024 - 65/1/1)



A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ .

On the basis of the above information, answer the following questions :

(i) Express θ in terms of height of the camera installed on the pole and x .

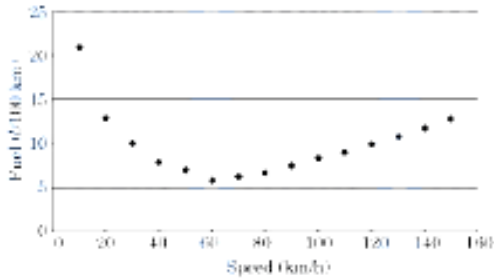
(ii) Find $d\theta/dx$.

(iii) (a) Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole.

(iii)(b) If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $3/101$ rad/s, then find the speed of the car.

Q.18 Over speeding increases fuel consumption and decreases fuel economy as a result of tyre rolling friction and air resistance. While vehicles reach optimal fuel economy at different speeds, fuel mileage usually decreases rapidly at speeds above 80 km/h.

(4 Mark) (CBSE 2024 - 65/2/1)



The relation between fuel consumption F (l/100 km) and speed V (km/h) under some constraints is given as

$$F = \frac{V^2}{500} - \frac{V}{4} + 14.$$

On the basis of the above information, answer the following questions :

- (i) Find F , when $V = 40$ km/h.
- (ii) Find $\frac{dF}{dV}$.
- (iii) (a) Find the speed V for which fuel consumption F is minimum.
- (iii)(b) Find the quantity of fuel required to travel 600 km at the speed V at which $\frac{dF}{dV} = -0.01$.

Q.19 It is given that function $f(x) = x^4 - 62x^2 + ax + 9$ attains local maximum value at $x = 1$. Find the value of 'a', hence obtain all other points where the given function $f(x)$ attains local maximum or local minimum values.

(5 Mark) (CBSE 2024 - 65/3/1)

Q.20 The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of cylinder so formed is maximum.

(5 Mark) (CBSE 2024 - 65/3/1)

Answer

Q.1 D

Q.2 A

Q.3 A

Q.4 B

Q.5 C

Q.6 B

Q.7

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = \frac{(x+1)(x-1)}{x^2}$$

$$f'(x) = 0 \Rightarrow x = -1, 1$$

$$f''(x) = \frac{2}{x^3} \Rightarrow f''(-1) = -2 < 0$$

$\therefore -1$

is a point of local maximum

The local maximum value

$$= f(-1) = -2 = M \quad f''(1) = 2 > 0$$

The local minimum value

$$= f(1) = 2 = m$$

$$M - m = -4$$

Q.8

$$\begin{aligned} f'(x) &= e^x + e^{-x} + 1 - \frac{1}{1+x^2} \\ &= e^x + \frac{1}{e^x} + \frac{x^2}{1+x^2} > 0 \text{ for all } x \in R \end{aligned}$$

$\therefore f$ is strictly increasing over its domain R

Q.9

$$\begin{aligned}f'(x) &= 4x^3 - 12x^2 = 4x^2(x - 3) \\ \Rightarrow 4x^2(x - 3) &< 0 \text{ for } x < 3, x \neq 0 \\ \Rightarrow f'(x) &< 0 \text{ for } x < 3, x \neq 0\end{aligned}$$

Thus,

$$\begin{aligned}f(x) &= x^4 - 4x^3 + 10 \\ \text{is strictly decreasing on} \\ &(-\infty, 0) \cup (0, 3)\end{aligned}$$

Q.10

$$\begin{aligned}\text{Given, } \frac{dV}{dt} &= 6 \text{ cm}^3/\text{sec. Since, } V = x^3 \\ \frac{dV}{dt} = 3x^2 \frac{dx}{dt} &\Rightarrow 6 = 3x^2 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2}{x^2} \text{ cm/sec} \\ \text{Now, Surface Area} = S = 6x^2 &\Rightarrow \frac{dS}{dt} = 12x \frac{dx}{dt} = 3 \text{ cm}^2/\text{sec}\end{aligned}$$

Q.11

$$\begin{aligned}f'(x) &= 12x^2 - 36x + 27 \\ &= 3(2x - 3)^2 \geq 0 \text{ for all } x \in R\end{aligned}$$

$\therefore f$
is increasing on R .

Hence $f(x)$ does not have maxima or minima.

Q.12

$$\begin{aligned}\text{Let } C = 2\pi r, \text{ be the circumference of the circle, } \frac{d(\pi r^2)}{dt} &= 2 \Rightarrow \frac{dr}{dt} = \frac{1}{\pi} \text{ cm/sec} \\ \frac{dC}{dt} = 2\pi \frac{dr}{dt} = \frac{2}{r} &= \frac{2}{5} \text{ cm/sec at } r = 5 \text{ cm}\end{aligned}$$

Q.13

$$\begin{aligned}f(x) = \frac{\log x}{x} &\Rightarrow f'(x) = \frac{1 - \log x}{x^2}; x > 0 \\ \text{for strictly increasing/decreasing, put } f'(x) = 0 &\Rightarrow x = e \text{ for strictly increasing, } x \in (0, e) \text{ and} \\ \text{for strictly decreasing } x &\in (e, \infty)\end{aligned}$$

Q.14

$$f(x) = \frac{x}{2} + \frac{2}{x}; x \in [1, 2]$$

$$\Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2}$$

for absolute maximum / minimum, put $f'(x) = 0$

$$\Rightarrow x^2 = 4 \Rightarrow x = 2$$

Now,

$$f(1) = \frac{5}{2}$$

and

$$f(2) = 2$$

\therefore absolute maximum value

$$= 5/2$$

and absolute minimum value

$$= 2$$

Q.15

(i) Revenue by selling x items $= R(x) = x \cdot p(x) = 450x - \frac{x^2}{2} \frac{dR}{dx} = 450 - x$

For Maxima or Minima, $\frac{dR}{dx} = 0 \Rightarrow x = 450$

$$\frac{d^2R}{dx^2} = -1 < 0$$

(Revenue is maximum when $x = 450$ units are sold)

(ii) At $x = 450$, $p = 450 - \frac{450}{2} = 225$

So, Rebate $= 350 - 225 =$ Rs. 125 per calculator

Q.16

(i) Let $A(x)$ be the area of the visiting card then,

$$\text{As } xy = 24, A(x) = (x + 3)(y + 2) = 2x + 3y + xy + 6 = 2x + \frac{72}{x} + 30$$

$$(ii) A'(x) = 2 - \frac{72}{x^2} \text{ and } A''(x) = \frac{144}{x^3},$$

solving $A'(x) = 0 \Rightarrow x = 6$ is the critical point.

$$A''(6) = \frac{144}{6^3} > 0, \therefore \text{Area of the card is minimum at } x = 6, y = 4$$

The dimension of the card with minimum area is Length = 9 cm, Breadth = 6 cm

Q.17

(i)

$$\tan \theta = \frac{5}{x} \Rightarrow \theta = \tan^{-1} \left(\frac{5}{x} \right)$$

(ii)

$$\frac{d\theta}{dx} = \frac{-5}{5^2 + x^2}$$

(iii)

$$(a) \left. \frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{-5}{5^2 + x^2} \times 20 \right]_{x=50}$$
$$= \frac{-100}{2525} \text{ or } \frac{-4}{101} \text{ rad/s}$$

(iii)(b)

(b)

$$\left. \frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} \Rightarrow \frac{3}{101} = \frac{-5}{5^2 + x^2} \right]_{x=50} \times \frac{dx}{dt}$$
$$\Rightarrow \frac{3}{101} = \frac{-5}{2525} \times \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -15 \text{ m/s}$$

Hence the speed is 15m/s

Q.18

(i) When

$$V = 40 \text{ km/h, } F = 36/5 \ell / 100 \text{ km}$$

(ii)

$$\frac{dF}{dV} = \frac{V}{250} - \frac{1}{4}$$

(iii)(a)

$$\frac{dF}{dV} = 0$$

$$\implies V = 62.5 \text{ km/h}$$

$$\frac{d^2F}{dV^2} = \frac{1}{250} > 0$$

at

$$V = 62.5 \text{ km/h}$$

Hence, F is minimum when

$$V = 62.5 \text{ km/h}$$

(iii) (b)

$$\frac{dF}{dV} = -0.01$$

$$\implies \frac{V}{250} - \frac{1}{4} = \frac{-1}{100}$$

$$\implies V = 60 \text{ km/h}$$

$$F = \frac{60^2}{500} - \frac{60}{4} + 14 = 6.2\ell/100 \text{ km}$$

Quantity of fuel required for 600 km

$$= 6.2 \times 6 = 37.2\ell$$

Q.19

$$f(x) = x^4 - 62x^2 + ax + 9 \implies f'(x) = 4x^3 - 124x + a$$

as at

$$x = 1, f$$

attains local maximum value,

$$f'(1) = 0 \implies a = 120$$

now,

$$f'(x) = 4x^3 - 124x + 120 = 4(x-1)(x^2 + x - 30) = 4(x-1)(x-5)(x+6)$$

Critical points are

$$x = -6, 1, 5$$

$$f''(x) = 12x^2 - 124$$

$$f''(-6) > 0, f''(1) < 0, f''(5) > 0$$

so

f

attains local maximum value at

$$x = 1$$

and local minimum value at

$$x = -6, 5$$

Q.20

Let length of rectangle be

x cm

and breadth be

$$(150 - x) \text{ cm}$$

Let

r

be the radius of cylinder

$$\Rightarrow 2\pi r = x \Rightarrow r = \frac{x}{2\pi}$$

$$V = \pi r^2 h = \pi \left(\frac{x^2}{4\pi^2} \right) (150 - x) = \frac{75x^2}{2\pi} - \frac{x^3}{4\pi}$$

$$\frac{dV}{dx} = \frac{150x}{2\pi} - \frac{3x^2}{4\pi}$$

$$\frac{dV}{dx} = 0 \Rightarrow x = 100 \text{ cm}$$

$$\left. \frac{d^2V}{dx^2} \right|_{x=100 \text{ cm}} = -\frac{75}{\pi} < 0 \Rightarrow V \text{ is maximum when } x = 100 \text{ cm}$$

Length of rectangle is 100 cm and breadth of rectangle is 50 cm.

Previous Years' CBSE Board Questions

6.2 Rate of Change of Quantities

VSA (1 mark)

- The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of _____ cm²/s. (2020) (Ap)
- The rate of change of the area of a circle with respect to its radius r , when $r = 3$ cm, is _____. (2020)

SA I (2 marks)

- The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. (2018) (Ap)
- The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. (Delhi 2017)
- The volume of a cube is increasing at the rate of 9 cm³/s. How fast is its surface area increasing when the length of an edge is 10 cm? (NCERT, AI 2017) (Ev)

LA I (4 marks)

- A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? (AI 2019)
- The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm? (Delhi 2015) (Ev)
- The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases, when the side is 10 cm. (AI 2014C)

6.3 Increasing and Decreasing Functions

MCQ

- The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is
(a) $(-1, \infty)$ (b) $(-2, -1)$ (c) $(-\infty, -2)$ (d) $[-1, 1]$ (2023)
- The function $f(x) = x^3 + 3x$ is increasing in interval
(a) $(-\infty, 0)$ (b) $(0, \infty)$ (c) R (d) $(0, 1)$ (2023)
- The interval, in which function $y = x^3 + 6x^2 + 6$ is increasing, is
(a) $(-\infty, -4) \cup (0, \infty)$ (b) $(-\infty, -4)$

- (c) $(-4, 0)$ (d) $(-\infty, 0) \cup (4, \infty)$
(Term I, 2021-22)

- The function $(x - \sin x)$ decreases for
(a) all x (b) $x < \frac{\pi}{2}$
(c) $0 < x < \frac{\pi}{4}$ (d) no value of x
(Term I, 2021-22)

VSA (1 mark)

- Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing. (2020)

SA I (2 marks)

- Find the interval in which the function $f(x) = 2x^3 - 3x$ is strictly increasing. (2023)
- Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R . (Delhi 2017)
- Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R . (AI 2017) (Ap)

LA I (4 marks)

- Find whether the function $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$; is increasing or decreasing in the interval $\frac{3\pi}{8} < x < \frac{5\pi}{8}$. (2019) (U)
- Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is
(a) strictly increasing
(b) strictly decreasing (2018) (Ap)
- Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
(a) strictly increasing
(b) strictly decreasing (Delhi 2014)
- Find the value(s) of x for which $y = [x(x-2)]^2$ is an increasing function. (AI 2014)
- Find the intervals in which the function $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is
(i) strictly increasing
(ii) strictly decreasing (Foreign 2014) (Ap)
- Find the intervals in which the function $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is
(a) strictly increasing
(b) strictly decreasing. (NCERT, AI 2014C) (Ev)

LA II (5/6 marks)

23. Find the intervals on which the function $f(x) = (x - 1)^3 (x - 2)^2$ is (a) strictly increasing (b) strictly decreasing. (2020)
24. Find the intervals in which the function f defined as $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$ is strictly increasing or decreasing. (2020) (U)
25. Find the intervals in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. (Delhi 2016) (An)
26. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is (i) strictly increasing (ii) strictly decreasing. (Delhi 2014C)

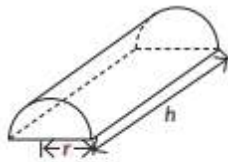
6.4 Maxima and Minima

MCQ

27. The value of x for which $(x - x^2)$ is maximum, is (a) $3/4$ (b) $1/2$ (c) $1/3$ (d) $1/4$ (Term I, 2021-22) (U)
28. A wire of length 20 cm is bent in the form of a sector of a circle. The maximum area that can be enclosed by the wire is (a) 20 sq.cm (b) 25 sq.cm (c) 10 sq.cm (d) 30 sq.cm (Term I, 2021-22)

Case study-Some young entrepreneur started a industry "young achievers" for casting metal into various shapes. They put up an advertisement online stating the same and expecting order to cast metal for toys, sculptures, decorative pieces and more.

A group of friends wanted to make innovative toys and hence contacted the "young achievers" to order them to cast metal into solid half cylinders with a rectangular base and semi-circular ends.



Based on the above information, answer the following questions (29 to 33):

29. The volume (V) of the casted half cylinder will be (a) $\pi r^2 h$ (b) $\frac{1}{3} \pi r^2 h$ (c) $\frac{1}{2} \pi r^2 h$ (d) $\pi r^2 (r + h)$ (Term I, 2021-22)
30. The total surface area (S) of the casted half cylinder will be (a) $\pi r h + 2\pi r^2 + r h$ (b) $\pi r h + \pi r^2 + 2r h$ (c) $2\pi r h + \pi r^2 + 2r h$ (d) $\pi r h + \pi r^2 + r h$ (Term I, 2021-22) (Ev)
31. The total surface area S can be expressed in terms of V and r as (a) $2\pi r + \frac{2V(\pi+2)}{\pi r}$ (b) $\pi r + \frac{2V}{\pi r}$ (c) $\pi r^2 + \frac{2V(\pi+2)}{\pi r}$ (d) $2\pi r^2 + \frac{2V(\pi+2)}{\pi r}$ (Term I, 2021-22) (Ap)

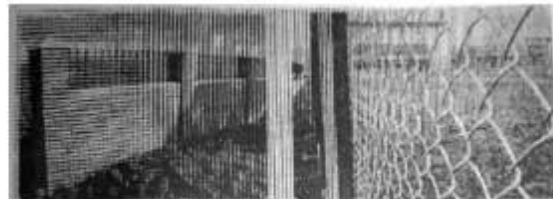
32. For the given half-cylinder of volume V , the total surface area S is minimum, when (a) $(\pi + 2)V = \pi^2 r^3$ (b) $(\pi + 2)V = \pi^2 r^2$ (c) $2(\pi + 2)V = \pi^2 r^3$ (d) $(\pi + 2)V = \pi^2 r$ (Term I, 2021-22)
33. The ratio $h : 2r$ for which S to be minimum will be equal to (a) $2\pi : \pi + 2$ (b) $2\pi : \pi + 1$ (c) $\pi : \pi + 1$ (d) $\pi : \pi + 2$ (Term I, 2021-22) (Cr)

VSA (1 mark)

34. The absolute minimum value of $f(x) = 2 \sin x$ in $\left[0, \frac{3\pi}{2}\right]$ is _____. (2020)
35. The least value of the function $f(x) = ax + \frac{b}{x}$ ($a > 0, b > 0, x > 0$) is _____. (2020)

LAI (4 marks)

36. **Case-study** : Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.



Based on the above information, answer the following questions :

- (i) Let ' x ' metres denote the length of the side of the garden perpendicular to the brick wall and ' y ' metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden.
- (ii) Determine the maximum value of $A(x)$. (2023)
37. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question? (2018) (Ap)

LA II (5 / 6 marks)

38. The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. (2023)
39. Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. (2023)
40. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right

circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. (2020)

41. Find the minimum value of $(ax + by)$, where $xy = c^2$. (2020, Foreign 2015)
42. Amongst all open (from the top) right circular cylindrical boxes of volume $125\pi \text{ cm}^3$, find the dimensions of the box which has the least surface area. (2020)
43. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume. (2020) (Ap)
44. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. (2020C)
45. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. (2019)
46. Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base. (2019)
47. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank? (NCERT, Delhi 2019) (Cr)
48. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (AI 2019)
49. Show that the right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base. (2019C)
50. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. (NCERT Exemplar, Delhi 2017, AI 2016, 2014)
51. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. (AI 2017) (Ev)
52. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere. (Delhi 2016, AI 2014) (Ap)
53. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. (AI 2016)
54. The sum of the surface areas of a cuboid with sides x , $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x

is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes.

(Foreign 2016) (Ap)

OR

The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}$, x and $2x$, is constant. Show that the

sum of their volumes is minimum if x is equal to three times the radius of sphere. (AI 2015C)

55. Find the local maxima and local minima of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values. (Delhi 2015)
56. Find the coordinates of a point of the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$. (Foreign 2015)
57. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 m^3 . If building of tank costs ₹ 100 per square metre for the base and ₹ 50 per square metre for the sides, find the cost of least expensive tank. (Delhi 2015C) (Ap)
58. A point on the hypotenuse of a right triangle is at distance 'a' and 'b' from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$. (NCERT, Delhi 2015C)
59. Of all the closed right circular cylindrical cans of volume $128\pi \text{ cm}^3$, find the dimensions of the can which has minimum surface area. (Delhi 2014)
60. Show that the semi vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1} \frac{1}{\sqrt{3}}$. (Delhi 2014) (Ev)
61. Prove that the semi vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1} \sqrt{2}$. (Delhi 2014)
62. The sum of the perimeters of a circle and a square is k , where k is some constant. Prove that the sum of their areas is least when the side of the square is equal to the diameter of the circle. (Foreign 2014, Delhi 2014C)
63. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base. (Foreign 2014) (Ap)
64. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10 m. Find the dimension of the window to admit maximum light through the whole opening. (Foreign 2014)
65. AB is a diameter of a circle and C is any point on the circle. Show that the area of $\triangle ABC$ is maximum, when it is isosceles. (AI 2014C)
66. Find the point P on the curve $y^2 = 4ax$ which is nearest to the point $(11a, 0)$. (AI 2014C) (Ev)
67. If the length of three sides of a trapezium other than base is 10 cm each, then find the area of the trapezium when it is maximum. (NCERT, AI 2014C) (Ap)

6.2 Rate of Change of Quantities

SA I (2 marks)

1. A man 1.6 m tall walks at the rate of 0.3 m/sec away from a street light that is 4 m above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow lengthening? (2022-23)

6.3 Increasing and Decreasing Functions

MCQ

2. Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is strictly increasing.
 (a) $(-\infty, 2) \cup (2, \infty)$ (b) $(2, \infty)$
 (c) $(-\infty, 2)$ (d) $(-\infty, 2] \cup (2, \infty)$

(Term I, 2021-22) (Ev)

3. The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
 (a) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$
 (b) Strictly decreasing in $(-2, 3)$
 (c) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$
 (d) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$

(Term I, 2021-22)

4. The value of b for which the function $f(x) = x + \cos x + b$ is strictly decreasing over R is
 (a) $b < 1$ (b) No value of b exists
 (c) $b \leq 1$ (d) $b \geq 1$

(Term I, 2021-22) (Ap)

SA II (3 marks)

5. Find the intervals in which the function f given by $f(x) = \tan x - 4x$, $x \in \left(0, \frac{\pi}{2}\right)$ is
 (a) strictly increasing (b) strictly decreasing
 (2020-21)

6.4 Maxima and Minima

MCQ

6. The least value of the function $f(x) = 2\cos x + x$ in the closed interval $\left[0, \frac{\pi}{2}\right]$ is
 (a) 2 (b) $\frac{\pi}{6} + \sqrt{3}$
 (c) $\frac{\pi}{2}$
 (d) The least value does not exist.

(Term I, 2021-22)

7. The area of a trapezium is defined by function f and given by $f(x) = (10+x)\sqrt{100-x^2}$, then the area when it is maximised is

- (a) 75 cm^2 (b) $7\sqrt{3} \text{ cm}^2$
 (c) $75\sqrt{3} \text{ cm}^2$ (d) 5 cm^2

(Term I, 2021-22) (Gr)

8. The maximum value of $[x(x-1)+1]^{1/3}$, $0 \leq x \leq 1$ is

- (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt[3]{\frac{1}{3}}$

(Term I, 2021-22)

Case Study : The fuel cost per hour for running a train is proportional to the square of the speed it generates in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges to run the train amount is ₹ 1200 per hour.



Assume the speed of the train as v km/h.

Based on the given information, answer the following questions (9 to 13).

9. Given that the fuel cost per hour is k times the square of the speed the train generates in km/h, the value of k is

- (a) $\frac{16}{3}$ (b) $\frac{1}{3}$ (c) 3 (d) $\frac{3}{16}$

10. If the train has travelled a distance of 500 km, then the total cost of running the train is given by function

- (a) $\frac{15}{16}v + \frac{600000}{v}$ (b) $\frac{375}{4}v + \frac{600000}{v}$
 (c) $\frac{5}{16}v^2 + \frac{150000}{v}$ (d) $\frac{3}{16}v + \frac{6000}{v}$

11. The most economical speed to run the train is

- (a) 18 km/h (b) 5 km/h
 (c) 80 km/h (d) 40 km/h

12. The fuel cost for the train to travel 500 km at the most economical speed is

- (a) ₹ 3750 (b) ₹ 750
(c) ₹ 7500 (d) ₹ 75000

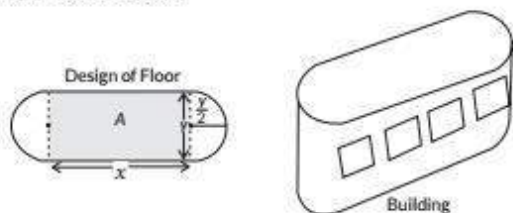
13. The total cost of the train to travel 500 km at the most economical speed is

- (a) ₹ 3750 (b) ₹ 75000
(c) ₹ 7500 (d) ₹ 15000

(Term I, 2021-22)

Case study based questions are compulsory. Attempt any 4 sub parts from question. Each sub-part carries 1 mark.

14. An architect designs a building for a multi-national company. The floor consists of a rectangular region with semicircular ends having a perimeter of 200 m as shown below:



Based on the above information answer the following:

(i) If x and y represents the length and breadth of the rectangular region, then the relation between the variables is

- (a) $x + \pi y = 100$ (b) $2x + \pi y = 200$
(c) $\pi x + y = 50$ (d) $x + y = 100$

(ii) The area of the rectangular region A expressed as a function of x is

- (a) $\frac{2}{\pi}(100x - x^2)$ (b) $\frac{1}{\pi}(100x - x^2)$
(c) $\frac{x}{\pi}(100 - x)$ (d) $\pi y^2 + \frac{2}{\pi}(100x - x^2)$

(iii) The maximum value of area A is

- (a) $\frac{\pi}{3200} \text{m}^2$ (b) $\frac{3200}{\pi} \text{m}^2$
(c) $\frac{5000}{\pi} \text{m}^2$ (d) $\frac{1000}{\pi} \text{m}^2$

(iv) The CEO of the multi-national company is interested in maximizing the area of the whole floor including the semi-circular ends. For this to happen the value of x should be

- (a) 0 m (b) 30 m
(c) 50 m (d) 80 m

(v) The extra area generated if the area of the whole floor is maximized is

- (a) $\frac{3000}{\pi} \text{m}^2$ (b) $\frac{5000}{\pi} \text{m}^2$

- (c) $\frac{7000}{\pi} \text{m}^2$

(d) No change, Both areas are equal (2020-21) **Cr**

LA I (4 marks)

15. Case-Study : Read the following passage and answer the questions given below.

In an elliptical sport field the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



(i) If the length and the breadth of the rectangular field be $2x$ and $2y$ respectively, then find the area function in terms of x .

(ii) Find the critical point of the function.

(iii) Use First Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area.

OR

Use Second Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area. (2022-23)

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. Let r be the radius and A be the area of circle.

Given that $\frac{dr}{dt} = 3 \text{ cm/sec}$... (i)

We know that, area of circle $A = \pi r^2$

$$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \cdot 3$$

[Using (i)]

$$= 6\pi r$$

$$\therefore \left(\frac{dA}{dt}\right)_{r=2 \text{ cm}} = 12\pi \text{ cm}^2/\text{s}$$

2. Let r be the radius and A be the area of circle.

We know that, area of circle, $A = \pi r^2$

$$\therefore \frac{dA}{dr} = 2\pi r \Rightarrow \left(\frac{dA}{dr}\right)_{r=3} = 6\pi \text{ cm}$$

3. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

$$\Rightarrow \frac{dC}{dx} = 0.015x^2 - 0.04x + 30$$

Now, $\left(\frac{dC}{dx}\right)_{x=3} = 0.015 \times 3^2 - 0.04 \times 3 + 30 = 30.015$

4. Let r , S and V respectively be the radius, surface area and volume of sphere at any time t .

Given, $\frac{dV}{dt} = 3 \text{ cm}^3/\text{sec}$

We know that, volume of sphere $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2} \text{ cm/sec}$$

We know that, surface area of sphere $S = 4\pi r^2$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi r \left(\frac{3}{4\pi r^2}\right) \Rightarrow \frac{dS}{dt} = \frac{6}{r}$$

$$\therefore \left(\frac{dS}{dt}\right)_{r=2\text{cm}} = \frac{6}{2} = 3 \text{ cm}^2/\text{sec}$$

5. Let l be the length of an edge and V be the volume of a cube respectively.

Given, $\frac{dV}{dt} = 9 \text{ cm}^3/\text{s}$ and $l = 10 \text{ cm}$

We know that, volume of cube (V) = l^3

$$\therefore \frac{dV}{dt} = \frac{d}{dt}(l^3) \Rightarrow 9 = 3l^2 \frac{dl}{dt}$$

$$\Rightarrow \frac{dl}{dt} = \frac{3}{l^2} \quad \dots(i)$$

And, surface area of cube (A) = $6l^2$

$$\therefore \frac{dA}{dt} = \frac{d}{dt}(6l^2) = 12l \frac{dl}{dt} = 12l \times \frac{3}{l^2} \quad \text{(From (i))}$$

$$= \frac{36}{l}$$

$$\therefore \left(\frac{dA}{dt}\right)_{l=10} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{s}$$

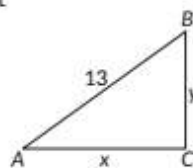
6. Let foot of the ladder is at a distance x m from the wall and height on the wall is y m.

Here, $x^2 + y^2 = (13)^2$ [Using Pythagoras theorem]

Differentiating with respect to t , we get

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-x}{y} \frac{dx}{dt}$$



When $x = 5 \text{ m}$, $y^2 = (13)^2 - (5)^2 = 169 - 25 = 144$

$$\therefore y = 12 \text{ m}$$

Also, $\frac{dx}{dt} = 2 \text{ cm/sec}$ [Given]

$$\therefore \frac{dy}{dt} = \frac{-5}{12} \times 2 = \frac{-5}{6} \text{ cm/sec}$$

7. Let ' a ' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2 \text{ cm/sec}$

Let ' A ' be the area of an equilateral triangle, then

$$A = \frac{\sqrt{3}}{4} a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4} a \frac{da}{dt} = \frac{\sqrt{3}}{2} a \frac{da}{dt}$$

$$\therefore \left(\frac{dA}{dt}\right)_{a=20} = \frac{\sqrt{3}}{2} \times 20 \times 2 = 20\sqrt{3} \text{ cm}^2/\text{sec}$$

8. Let ' a ' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2 \text{ cm/sec}$

Let ' A ' be the area of an equilateral triangle, then

$$A = \frac{\sqrt{3}}{4} a^2 \Rightarrow \frac{dA}{dt} = 2 \times \frac{\sqrt{3}}{4} a \frac{da}{dt} = \frac{\sqrt{3}}{2} a \frac{da}{dt}$$

$$\therefore \left(\frac{dA}{dt}\right)_{a=10} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{sec}$$

Concept Applied

Power rule of derivative: $\frac{d}{dx}(x^n) = nx^{n-1}$

9. (b): We have, $f(x) = 2x^3 + 9x^2 + 12x - 1$

$$\Rightarrow f'(x) = 6x^2 + 18x + 12$$

For decreasing, $f'(x) < 0$

$$\therefore 6x^2 + 18x + 12 < 0$$

$$\Rightarrow x^2 + 3x + 2 < 0 \Rightarrow (x+1)(x+2) < 0 \Rightarrow -2 < x < -1$$

So, $f(x)$ is decreasing, if $x \in (-2, -1)$.

10. (c): $f(x) = x^3 + 3x$

For increasing, we must have $f'(x) > 0$

$$\therefore f'(x) = 3x^2 + 3 > 0 \Rightarrow 3(x^2 + 1) > 0$$

$$\Rightarrow x^2 + 1 > 0, \text{ which is true } \forall x \in \mathbb{R}.$$

11. (a): Given, $y = x^3 + 6x^2 + 6 \Rightarrow \frac{dy}{dx} = 3x^2 + 12x$

For increasing, $\frac{dy}{dx} > 0 \Rightarrow 3x^2 + 12x > 0 \Rightarrow 3x(x+4) > 0$



So, y is strictly increasing in $(-\infty, -4) \cup (0, \infty)$.

12. (d): Let $f(x) = x - \sin x$

Differentiating w.r.t. x , we get $f'(x) = 1 - \cos x$

For function to be decreasing, $f'(x) < 0$

$$\Rightarrow 1 - \cos x < 0 \Rightarrow \cos x > 1,$$

which is not possible, because maximum value of $\cos x$ is 1.

$\therefore f(x) = (x - \sin x)$ doesn't decrease at any value of x .

13. Let $y = f(x) = 7 - 4x - x^2$

$$\therefore \frac{dy}{dx} = -4 - 2x$$

For strictly increasing, $\frac{dy}{dx} > 0$

$$\Rightarrow -4 - 2x > 0 \Rightarrow x < -2$$

\therefore Required interval is $(-\infty, -2)$.

14. Let $y = f(x) = 2x^3 - 3x \therefore \frac{dy}{dx} = 6x^2 - 3$

For strictly increasing, $\frac{dy}{dx} > 0$

$$\Rightarrow 6x^2 - 3 > 0 \Rightarrow x^2 > \frac{1}{2}$$

So, $f(x)$ is strictly increasing in $x \in \left(-\infty, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$.

15. We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$$\Rightarrow f'(x) = 12x^2 - 36x + 27$$

$$= 12 \left(x^2 - 3x + \frac{9}{4} - \frac{9}{4} \right) + 27$$

$$= 12 \left(x - \frac{3}{2} \right)^2 - 27 + 27 = 12 \left(x - \frac{3}{2} \right)^2 \geq 0 \quad \forall x \in \mathbb{R}$$

Hence, $f(x)$ is always increasing on \mathbb{R} .

16. We have, $f(x) = x^3 - 3x^2 + 6x - 100$
Differentiating (i) w.r.t. x , we get

$$f'(x) = 3x^2 - 6x + 6$$

$$= 3(x^2 - 2x + 1) + 3 = 3(x - 1)^2 + 3 > 0$$

(\because For all values of x , $(x - 1)^2$ is always positive)

$$\therefore f'(x) > 0$$

So, $f(x)$ is an increasing function on \mathbb{R} .

Answer Tips

\Rightarrow If $f'(x) > 0 \Rightarrow f$ is strictly increasing function.

17. We have, $f(x) = \cos \left(2x + \frac{\pi}{4} \right)$

$$\therefore f'(x) = -2 \sin \left(2x + \frac{\pi}{4} \right)$$

$$\text{Given, } \frac{3\pi}{8} < x < \frac{5\pi}{8}$$

$$\Rightarrow \frac{3\pi}{4} < 2x < \frac{5\pi}{4} \Rightarrow \frac{\pi}{4} + \frac{3\pi}{4} < 2x + \frac{\pi}{4} < \frac{5\pi}{4} + \frac{\pi}{4}$$

$$\Rightarrow \pi < 2x + \frac{\pi}{4} < \frac{3\pi}{2} \Rightarrow \sin \left(2x + \frac{\pi}{4} \right) < 0$$

[\because sin function is negative in IIIrd and IVth quadrant]

$$\Rightarrow -2 \sin \left(2x + \frac{\pi}{4} \right) > 0$$

$$\Rightarrow f'(x) > 0$$

Hence, $f(x)$ is increasing in $\left(\frac{3\pi}{8}, \frac{5\pi}{8} \right)$.

18. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$... (i)

$f(x)$ being polynomial function is continuous and derivable on \mathbb{R} .

Differentiating (i) w.r.t. x , we get

$$f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$$= x^3 - 3x^2 - 10x + 24$$

$$= (x - 2)(x^2 - x - 12) = (x - 2)(x - 4)(x + 3)$$

(a) For strictly increasing, $f'(x) > 0$

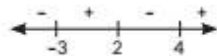
$$\Rightarrow (x - 2)(x - 4)(x + 3) > 0$$

$$\Rightarrow x \in (-3, 2) \cup (4, \infty)$$

(b) For strictly decreasing, $f'(x) < 0$

$$\Rightarrow (x - 2)(x - 4)(x + 3) < 0$$

$$\Rightarrow x \in (-\infty, -3) \cup (2, 4)$$



19. We have, $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$$

$$\Rightarrow f'(x) = 12x(x + 1)(x - 2)$$

Now, $f'(x) = 0$

$$\Rightarrow 12x(x + 1)(x - 2) = 0$$

$$\Rightarrow x = -1, x = 0 \text{ or } x = 2$$

Hence, these points divide the whole real line into four disjoint open intervals namely $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -1)$	$(-)(-)(-) < 0$	Strictly decreasing
$(-1, 0)$	$(-)(+)(-) > 0$	Strictly increasing
$(0, 2)$	$(+)(+)(-) < 0$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) > 0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$.

(b) $f(x)$ is strictly decreasing in $(-\infty, -1) \cup (0, 2)$.

20. Here, $y = [x(x - 2)]^2 = x^2(x - 2)^2$

$$\Rightarrow \frac{dy}{dx} = 2x(x - 2)^2 + 2x^2(x - 2)$$

$$= 2x(x - 2)(x - 2 + x) = 4x(x - 1)(x - 2)$$

For y to be an increasing function, $\frac{dy}{dx} \geq 0$

$$\Rightarrow x(x - 1)(x - 2) \geq 0$$

Case 1: When $-\infty < x \leq 0$

$$\frac{dy}{dx} \leq 0 \Rightarrow y \text{ is a decreasing function.}$$

Case 2: When $0 \leq x \leq 1$

$$\frac{dy}{dx} \geq 0 \Rightarrow y \text{ is an increasing function.}$$

Case 3: When $1 \leq x \leq 2$

$$\frac{dy}{dx} \leq 0 \Rightarrow y \text{ is a decreasing function.}$$

Case 4: When $2 \leq x < \infty$

$$\frac{dy}{dx} \geq 0 \Rightarrow y \text{ is an increasing function.}$$

$\therefore y$ is an increasing function in $[0, 1] \cup [2, \infty)$

21. We have, $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$... (i)

$f(x)$ being a polynomial function is continuous and derivable on \mathbb{R} .

Differentiating (i) w.r.t. x , we get

$$f'(x) = \frac{3}{2} \times 4x^3 - 12x^2 - 90x$$

$$\Rightarrow f'(x) = 6x^3 - 12x^2 - 90x = 6x(x^2 - 2x - 15)$$

$$= 6x(x - 5)(x + 3)$$

(i) For strictly increasing, $f'(x) > 0$

$$\Rightarrow 6x(x - 5)(x + 3) > 0$$

$$\Rightarrow x \in (-3, 0) \cup (5, \infty)$$

(ii) For strictly decreasing, $f'(x) < 0$

$$\Rightarrow 6x(x - 5)(x + 3) < 0$$

$$\Rightarrow x \in (-\infty, -3) \cup (0, 5)$$

Answer Tips

\Rightarrow If $f'(x) < 0 \Rightarrow f$ is strictly decreasing function.

\Rightarrow If $f'(x) > 0 \Rightarrow f$ is strictly increasing function.

22. Here, $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$

Differentiating w.r.t. x , we get

$$f'(x) = \frac{3}{10} \cdot 4x^3 - \frac{4}{5} \cdot 3x^2 - 3 \cdot 2x + \frac{36}{5} \cdot 1$$

$$= \frac{6}{5}(x^3 - 2x^2 - 5x + 6) = \frac{6}{5}(x-1)(x^2 - x - 6)$$

$$= \frac{6}{5}(x-1)(x+2)(x-3)$$

$$\therefore f'(x) = 0 \Rightarrow x = -2, 1, 3.$$

Hence, the points divide the real line into four disjoint intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -2)$	$(-)(-)(-) < 0$	Strictly decreasing
$(-2, 1)$	$(-)(+)(-) > 0$	Strictly increasing
$(1, 3)$	$(+)(+)(-) < 0$	Strictly decreasing
$(3, \infty)$	$(+)(+)(+) > 0$	Strictly increasing

- (a) $f(x)$ is strictly increasing in $(-2, 1) \cup (3, \infty)$.
 (b) $f(x)$ is strictly decreasing in $(-\infty, -2) \cup (1, 3)$.

23. We have, $f(x) = (x-1)^3(x-2)^2$

Differentiating equation (1) w.r.t. x , we get

$$f'(x) = (x-1)^3 \frac{d}{dx}(x-2)^2 + (x-2)^2 \frac{d}{dx}(x-1)^3$$

$$= (x-1)^2(x-2)[2(x-1) + 3(x-2)]$$

$$= (x-1)^2(x-2)(2x-2+3x-6)$$

$$\Rightarrow f'(x) = (x-1)^2(x-2)(5x-8)$$

Now put $f'(x) = 0$

$$\Rightarrow (x-1)^2(x-2)(5x-8) = 0 \Rightarrow x = 1, 2, 8/5$$

The points divide the real line into four disjoint intervals

$(-\infty, 1)$, $(1, 8/5)$, $(8/5, 2)$ and $(2, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, 1)$	$(+)(-)(-) > 0$	Strictly increasing
$(1, 8/5)$	$(+)(-)(-) > 0$	Strictly increasing
$(8/5, 2)$	$(+)(-)(+) < 0$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) > 0$	Strictly increasing

- (a) $f(x)$ is strictly increasing in $(-\infty, 1) \cup (1, 8/5) \cup (2, \infty)$.
 (b) $f(x)$ is strictly decreasing in $(8/5, 2)$.

24. The given function is

$$f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$$

$$\Rightarrow f'(x) = \cos x - \sin x$$

$$\text{Now } f'(x) = 0 \Rightarrow \cos x - \sin x = 0$$

$$\Rightarrow \tan x = 1$$

$$\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

The points $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ divide the interval $[0, 2\pi]$ into

three disjoint intervals, $[0, \pi/4)$, $(\pi/4, 5\pi/4)$, $(\frac{5\pi}{4}, 2\pi]$

$$\text{Now } f'(x) > 0 \text{ in } \left[0, \frac{\pi}{4}\right)$$

$$\therefore f \text{ is strictly increasing in } \left[0, \frac{\pi}{4}\right).$$

$$f'(x) < 0 \text{ in } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

$$\therefore f \text{ is strictly decreasing in } \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

$$\text{and } f'(x) > 0 \text{ in } \left(\frac{5\pi}{4}, 2\pi\right]$$

$$\therefore f \text{ is strictly increasing in } \left(\frac{5\pi}{4}, 2\pi\right].$$

Thus, the function f is strictly increasing in

$$\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right].$$

25. $f(x) = \sin 3x - \cos 3x$

$$\Rightarrow f'(x) = 3 \cos 3x + 3 \sin 3x$$

$$f'(x) = 0 \Rightarrow 3 \cos 3x = -3 \sin 3x$$

$$\Rightarrow \cos 3x = -\sin 3x \Rightarrow \tan 3x = -1$$

$$\text{which gives } 3x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4} \text{ or } \frac{11\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{4} \text{ or } \frac{7\pi}{12} \text{ or } \frac{11\pi}{12}$$

$$[\because 0 < x < \pi]$$

The points $x = \frac{\pi}{4}$, $x = \frac{7\pi}{12}$ and $x = \frac{11\pi}{12}$ divide the interval

$(0, \pi)$ into four disjoint intervals,

$$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$$

$$\text{Now, } f'(x) > 0 \text{ in } \left(0, \frac{\pi}{4}\right)$$

$$\Rightarrow f \text{ is strictly increasing in } \left(0, \frac{\pi}{4}\right)$$

$$f'(x) < 0 \text{ in } \left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$$

$$\Rightarrow f \text{ is strictly decreasing in } \left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$$

$$f'(x) > 0 \text{ in } \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$$

$$\Rightarrow f \text{ is strictly increasing in } \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$$

$$f'(x) < 0 \text{ in } \left(\frac{11\pi}{12}, \pi\right)$$

$$\Rightarrow f \text{ is strictly decreasing in } \left(\frac{11\pi}{12}, \pi\right)$$

Hence, f is strictly increasing in the intervals

$$\left(0, \frac{\pi}{4}\right) \cup \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$$

and f is strictly decreasing in the intervals

$$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$$

26. Here, $f(x) = x^2 - x + 1; x \in (-1, 1)$

$$\Rightarrow f'(x) = 2x - 1$$

$$f'(x) = 0 \Rightarrow x = \frac{1}{2}$$

$$\text{Now } f'(x) = 2\left(x - \frac{1}{2}\right) > 0 \text{ for } \frac{1}{2} < x < 1$$

$$\Rightarrow f \text{ is strictly increasing in } \left(\frac{1}{2}, 1\right)$$

Also $f'(x) = 2\left(x - \frac{1}{2}\right) < 0$ for $-1 < x < \frac{1}{2}$

$\Rightarrow f$ is strictly decreasing in $\left(-1, \frac{1}{2}\right)$.

Thus f is neither increasing nor decreasing in $(-1, 1)$.

27. (b): Let $f(x) = x - x^2$


$\therefore f'(x) = 1 - 2x$

For critical point, $f'(x) = 0$

$\Rightarrow 1 - 2x = 0 \Rightarrow x = 1/2$

Now, at $x = 1/2$, $f''(x) = -2 < 0$

So, $f(x)$ has maximum value at $x = 1/2$.

Concept Applied 

$\Rightarrow x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.
The value of $f(c)$ is local maximum value of f .

28. (b): Let r be the radius, θ be the central angle and l be the length of the circular sector.

Given, $l + 2r = 20$

$\Rightarrow r\theta + 2r = 20$ ($\because l = r\theta$) $\Rightarrow \theta = \frac{20-2r}{r}$

Let A be the area of the circular sector.

$\therefore A = \frac{\pi r^2 \theta}{2\pi} = \frac{r^2}{2} \cdot \left(\frac{20-2r}{r}\right) = r(10-r)$

$\Rightarrow \frac{dA}{dr} = 10 - 2r$

For maximum or minimum value of A , we have

$\frac{dA}{dr} = 0 \Rightarrow r = 5$ and $\frac{d^2A}{dr^2} = -2 < 0$

\therefore Area is maximum at $r = 5$

\therefore Maximum area, $A = 5(10 - 5) = 25 \text{ cm}^2$

29. (c): \because Volume of cylinder = $\pi r^2 h$

$\therefore V =$ Volume of casted half cylinder = $(1/2)\pi r^2 h$

30. (b): Total surface area, $S = \frac{2\pi r(r+h)}{2} + 2rh$
 $= \pi r^2 + \pi rh + 2rh$

31. (c): Here, $S = \pi r^2 + \frac{2V(\pi+2)}{\pi r}$ [$\because V = \frac{1}{2}\pi r^2 h \Rightarrow \frac{2V}{\pi r} = rh$]

32. (a): $\because S = \pi r^2 + \frac{2V(\pi+2)}{\pi r}$
 $\Rightarrow \frac{dS}{dr} = 2\pi r - \frac{2V(\pi+2)}{\pi} \times \frac{1}{r^2}$

For S to be minimum, $\frac{dS}{dr} = 0$

$\Rightarrow 2\pi r = \frac{2V(\pi+2)}{\pi r^2} \Rightarrow \pi^2 r^3 = V(\pi+2)$

33. (d): $\because V = \frac{1}{2}\pi r^2 h$

and S will be minimum, when $(\pi+2)V = \pi^2 r^3$

$\Rightarrow V = \frac{\pi^2 r^3}{\pi+2}$

From (i) and (ii), we get

$\Rightarrow \frac{1}{2}\pi r^2 h = \frac{\pi^2 r^3}{\pi+2} \Rightarrow \pi r^2 h (\pi+2) = 2\pi^2 r^3$

$\Rightarrow h(\pi+2) = 2\pi r \Rightarrow \frac{h}{2r} = \frac{\pi}{\pi+2}$

Thus, required ratio i.e., $h : 2r$ is $\pi : \pi+2$.

34. Here, $f(x) = 2\sin x$

$\Rightarrow f'(x) = 2\cos x$

Putting $f'(x) = 0$

$\Rightarrow 2\cos x = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore \frac{\pi}{2}, \frac{3\pi}{2}$ are the critical points

At $x = \frac{\pi}{2}$, $f(x) = 2 \times 1 = 2$

At $x = \frac{3\pi}{2}$, $f(x) = 2 \sin\left(\pi + \frac{\pi}{2}\right) = -2 \sin \frac{\pi}{2} = -2$

Hence, absolute minimum value of $f(x)$ is -2 .

35. We have, $f(x) = ax + \frac{b}{x}$

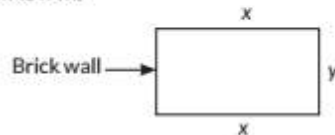
$\therefore f'(x) = a - (b/x^2)$

Putting $f'(x) = 0 \Rightarrow a - \frac{b}{x^2} = 0 \Rightarrow a = \frac{b}{x^2} \Rightarrow x = \sqrt{\frac{b}{a}}$ (as $x > 0$)

The least value of $f(x)$ is

$f\left(\sqrt{\frac{b}{a}}\right) = a\left(\sqrt{\frac{b}{a}}\right) + \sqrt{ab} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$

36. Given, the length of side of garden perpendicular to the brick wall is x m.



The length of the side parallel to the brick wall is y m.

(i) $2x + y = 200$

We know that area of rectangle is $l \times b$

$\Rightarrow A(x) = xy = x(200 - 2x) = 200x - 2x^2$

(ii) Since, $A(x) = 200x - 2x^2$

...(i)

Differentiating (i) w.r.t. x , we get

$\frac{d}{dx} A(x) = 200 - 4x$

...(ii)

For critical point $\frac{d}{dx} A(x) = 0$

$\Rightarrow 200 - 4x = 0 \Rightarrow 4x = 200 \Rightarrow x = 50$

Again differentiating (ii) w.r.t. x , we get

$\frac{d^2}{dx^2} A(x) = -4 < 0$ i.e., area $A(x)$ is maximum at $x = 50$

Hence, maximum area is

$A(50) = 200(50) - 2(50)^2$
 $= 10000 - 5000 = 5000 \text{ m}^2$

...(i)

37. Let x be the side of square base and y be the height of the open tank.

$\therefore l = x, b = x$ and $h = y$

where l, b and h be the length, breadth and height of tank respectively.

...(ii)

$$\text{Volume of tank } V = x^2y \Rightarrow y = \frac{V}{x^2}$$

The cost of the material will be least if the total surface area is least.

$$\text{Total surface area of tank (S)} = x^2 + 4xy$$

$$\Rightarrow S = x^2 + 4x\left(\frac{V}{x^2}\right) \quad \left(\because y = \frac{V}{x^2}\right)$$

$$\Rightarrow S = x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

$$\text{For maxima or minima, } \frac{dS}{dx} = 0$$

$$\Rightarrow 2x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = 2V \Rightarrow x = 2y \quad (\because V = x^2y)$$

$$\text{Also, } \frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0$$

$$\therefore \text{Cost of material is least, when } y = \frac{x}{2}$$

i.e., the depth of the tank is half of its width.

As the cost is borne by nearby settled lower income families it shows that they are spending money on social welfare so that no body will face the water problem in future. It shows social responsibility.

Concept Applied

$x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$.
The value of $f(c)$ is local maximum value of f .

38. Let the side of an equilateral triangle be $2x$ cm, then its median = $\sqrt{3}x$

$$\text{Let } M = \sqrt{3}x \Rightarrow \frac{dM}{dt} = \sqrt{3} \frac{dx}{dt}$$

$$\Rightarrow 2\sqrt{3} = \sqrt{3} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \text{ cm/sec}$$

Now, side of triangle = $2x$

$$\Rightarrow s = 2x$$

$$\Rightarrow \frac{ds}{dt} = 2 \frac{dx}{dt} = 2 \times 2 \text{ cm/sec} = 4 \text{ cm/sec}$$

So, side is increasing at the rate of 4 cm/sec.

39. Let the two numbers be x and y .

According to question, we have

$$x + y = 5 \Rightarrow y = 5 - x$$

$$\text{Let } p = x^3 + y^3$$

$$= x^3 + (5 - x)^3$$

$$= x^3 + 125 - x^3 - 75x + 15x^2$$

$$\Rightarrow p = 15x^2 - 75x + 125$$

Differentiating with respect to x , we get

$$\frac{dp}{dx} = 30x - 75$$

$$\text{For minimum, } \frac{dp}{dx} = 0 \Rightarrow 30x - 75 = 0 \Rightarrow x = \frac{5}{2}$$

$$\text{Now, } \frac{d^2p}{dx^2} = 30 > 0$$

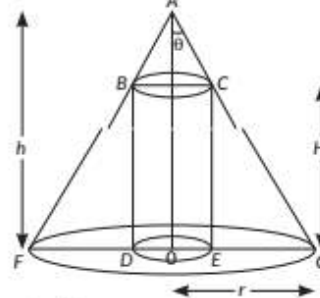
$$\text{So, } x^3 + y^3 \text{ is minimum at } x = \frac{5}{2}$$

$$\text{From (i), } y = 5 - \frac{5}{2} = \frac{5}{2}$$

$$\begin{aligned} \text{So, required value} &= x^2 + y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 \\ &= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} = \frac{25}{2} \end{aligned}$$

40. Given the right circular cone of fixed height h and semi-vertical angle θ . Let R be the radius of the base and H be the height of the right circular cylinder that can be inscribed in the right circular cone.

In the figure, $\angle GAO = \theta$, $OG = r$, $OA = h$, $OE = R$ and $CE = H$



We have, $r/h = \tan \theta$

$$\therefore r = h \tan \theta$$

...(1)

Since, $\triangle AOG$ and $\triangle CEG$ are similar.

$$\therefore \frac{AO}{OG} = \frac{CE}{EG} = \frac{CE}{OG - OE}$$

$$\therefore \frac{h}{r} = \frac{H}{r - R} \Rightarrow R = r\left(1 - \frac{H}{h}\right)$$

$$\text{Now, volume of cylinder (V)} = \pi R^2 H = \pi r^2 \left(1 - \frac{H}{h}\right)^2 H$$

$$\text{For maximum volume, } \frac{dV}{dH} = 0$$

$$\Rightarrow \pi r^2 \left[H \times 2 \left(1 - \frac{H}{h}\right) \times -\frac{1}{h} + \left(1 - \frac{H}{h}\right)^2 \right] = 0$$

$$\Rightarrow \pi r^2 \left[-\frac{4H}{h} + \frac{3H^2}{h^2} + 1 \right] = 0 \Rightarrow \pi r^2 \left(\frac{H}{h} - 1 \right) \left(\frac{3H}{h} - 1 \right) = 0$$

$$\Rightarrow H = \frac{h}{3} \quad [\because H \neq h]$$

So, height of cylinder = $\frac{1}{3}$ of height of cone

Also, maximum volume of cylinder

$$\dots (i) = \pi r^2 \left(1 - \frac{1}{3}\right)^2 \times \frac{h}{3} = \left(\frac{1}{3}\pi r^2 h\right) \times \frac{4}{9}$$

$$= \frac{4}{9} \text{ of volume of cone.}$$

41. Let $u = ax + by$, where $xy = c^2$

$$\Rightarrow u = ax + b\left(\frac{c^2}{x}\right)$$

...(i)

Differentiating w.r.t. x , we get

$$\frac{du}{dx} = a - \frac{bc^2}{x^2} \text{ and } \frac{d^2u}{dx^2} = \frac{2bc^2}{x^3}$$

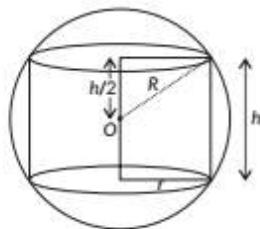
$$\text{For critical points, } \frac{du}{dx} = 0$$

$$\Rightarrow \frac{ax^2 - bc^2}{x^2} = 0 \Rightarrow x^2 = \frac{bc^2}{a}$$

45. Let r and h be the base radius and height of cylinder respectively.

$$\therefore \left(\frac{h}{2}\right)^2 + r^2 = R^2 \quad \dots (i)$$

Now, $V =$ Volume of the cylinder inscribed in a sphere
 $= \pi r^2 h$



$$\Rightarrow V = \pi h \left(R^2 - \frac{h^2}{4} \right) \quad \text{[Using (i)]}$$

$$\Rightarrow V = \pi \left(R^2 h - \frac{h^3}{4} \right)$$

Now differentiating w.r.t. h , we get

$$\frac{dV}{dh} = \pi \left(R^2 - \frac{3h^2}{4} \right) \quad \text{and} \quad \frac{d^2V}{dh^2} = \pi \left(0 - \frac{3}{4} \cdot 2h \right)$$

For maximum or minimum,

$$\frac{dV}{dh} = 0 \Rightarrow R^2 - \frac{3}{4}h^2 = 0 \Rightarrow h^2 = \frac{4}{3}R^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$$

For this value of h , $\frac{d^2V}{dh^2} = -\frac{3}{2} \pi \cdot \frac{2R}{\sqrt{3}} = -\sqrt{3}\pi R < 0$

$\Rightarrow V$ is maximum

Also maximum value of V

$$= \pi \cdot \frac{2R}{\sqrt{3}} \left(R^2 - \frac{1}{4} \cdot \frac{4}{3} R^2 \right) = \pi \cdot \frac{2R}{\sqrt{3}} \cdot \frac{2}{3} R^2 = \frac{4\pi}{3\sqrt{3}} R^3 \text{ cu. units}$$

Key Points

Volume of cylinder $= \pi r^2 h$

46. Let r be the radius of base of circular cylinder and h be its height. Let V be the volume and S be total surface area.

$$\therefore S = \pi r^2 + 2\pi r h \Rightarrow h = \frac{S - \pi r^2}{2\pi r} \quad \dots (i)$$

$$V = \pi r^2 h = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r} \right) \quad \text{[from (i)]}$$

$$\therefore V = \frac{1}{2} (S r - \pi r^3) \Rightarrow \frac{dV}{dr} = \frac{1}{2} (S - 3\pi r^2)$$

Put $\frac{dV}{dr} = 0 \Rightarrow \frac{1}{2} (S - 3\pi r^2) = 0 \Rightarrow r = \sqrt{\frac{S}{3\pi}}$

Now, $\frac{d^2V}{dr^2} = \frac{1}{2} (0 - 6\pi r) = -3\pi r$

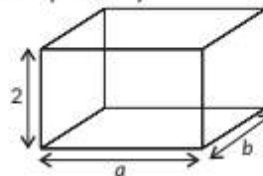
For $r = \sqrt{\frac{S}{3\pi}}$, $\frac{d^2V}{dr^2} = -3\pi \sqrt{\frac{S}{3\pi}} = -\sqrt{3\pi S} < 0$

\therefore Volume is greatest when $r = \sqrt{\frac{S}{3\pi}}$

$$\text{From (i), } h = \frac{S - \pi \frac{S}{3\pi}}{2\pi \sqrt{\frac{S}{3\pi}}} = \frac{\frac{2S}{3}}{2\sqrt{\frac{\pi S}{3}}} = \frac{S}{3} \times \frac{\sqrt{3}}{\sqrt{\pi S}} = \sqrt{\frac{S}{3\pi}} = r$$

Hence, proved.

47. Let a m and b m be the length and breadth of rectangular tank respectively.



\therefore Volume of tank $= 2ab = 8$ [Given]

$\Rightarrow ab = 4 \Rightarrow b = \frac{4}{a}$... (i)

If C is the total cost in rupees, then

$$C = 70(ab) + 45(2a + 2b) \times 2$$

$$\Rightarrow C = 70ab + 90(2a + 2b)$$

$$\Rightarrow C = 70(a) \left(\frac{4}{a} \right) + 180 \left(a + \frac{4}{a} \right) \quad \text{[Using (i)]}$$

$$\Rightarrow C = 280 + 180a + \frac{720}{a} \quad \dots (ii)$$

Differentiating (ii) w.r.t. ' a ', we get

$$\frac{dC}{da} = 180 - \frac{720}{a^2} \quad \text{and} \quad \frac{d^2C}{da^2} = \frac{720 \times 2}{a^3}$$

For maximum or minimum cost,

$$\frac{dC}{da} = 0 \Rightarrow 180 - \frac{720}{a^2} = 0 \Rightarrow a^2 = 4 \Rightarrow a = 2$$

For $a = 2$, $\frac{d^2C}{da^2} > 0 \Rightarrow C$ is least

Using (i), $a = 2$ and $b = 2$

Hence, cost of least expensive tank is

$$C = 280 + 360 + 360 = ₹ 1000$$

Key Points

Remember $\frac{d}{da} \left(\frac{1}{a} \right) = -\frac{1}{a^2}$

48. Let $ABCD$ be a rectangle inscribed in the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let $AB = 2q$, $DA = 2p$.

Then coordinates of A are (p, q) .

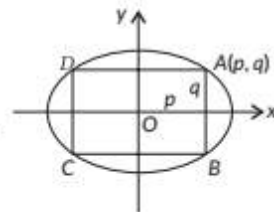
As A lies on the ellipse so

$$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \Rightarrow q^2 = b^2 \left(1 - \frac{p^2}{a^2} \right)$$

Now area A of the rectangle $ABCD = 2p \cdot 2q = 4pq$

$$\Rightarrow A^2 = 16p^2 q^2$$

$$= 16p^2 \cdot b^2 \left(1 - \frac{p^2}{a^2} \right) = 16b^2 \left(p^2 - \frac{p^4}{a^2} \right)$$



$$\therefore \frac{dA^2}{dp} = 16b^2 \left(2p - \frac{4p^3}{a^2} \right) \text{ and}$$

$$\frac{d^2A^2}{dp^2} = 16b^2 \left(2 - \frac{12p^2}{a^2} \right)$$

For A to be max. or min.

$$\frac{dA^2}{dp} = 0 \Rightarrow 2p - \frac{4p^3}{a^2} = 0 \Rightarrow p^2 = \frac{a^2}{2} \quad (\because p \neq 0)$$

For this value of p^2 ,

$$\frac{d^2A^2}{dp^2} = 16b^2 \left(2 - 12 \times \frac{1}{2} \right) = 16b^2(-4) < 0$$

Hence, A^2 is max. $\Rightarrow A$ is max.

\Rightarrow The area of the greatest rectangle inscribed in the

$$\text{ellipse} = 4pq = 4 \cdot \sqrt{\frac{a^2}{2}} \cdot \sqrt{b^2 \left(1 - \frac{1}{2} \right)} = 2ab \text{ sq. units}$$

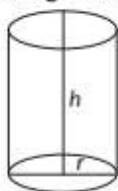
49. Let 'r' and 'h' be the radius and height of right circular cylinder.

Surface area of cylinder is given by

$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow 2\pi rh = S - 2\pi r^2$$

$$\text{or } h = \frac{S - 2\pi r^2}{2\pi} \quad \dots(i)$$



Volume of cylinder is given by

$$V = \pi r^2 h \quad \dots(ii)$$

On substituting the value of h from (i) into (ii), we get

$$V = \frac{Sr}{2} - \pi r^3 \quad \dots(iii)$$

On differentiating (iii) with respect to 'r', we get

$$\text{For maximum and minimum value, put } \frac{dV}{dr} = 0$$

$$\Rightarrow \frac{S}{2} - 3\pi r^2 = 0$$

$$\Rightarrow S = 6\pi r^2$$

$$\text{or } r^2 = \frac{S}{6\pi} \Rightarrow r = \sqrt{\frac{S}{6\pi}}$$

Again differentiating (iv) w.r.t. r, we have

$$\frac{d^2V}{dr^2} = -6\pi r$$

$$\left(\frac{d^2V}{dr^2} \right)_{r=\sqrt{\frac{S}{6\pi}}} = -6\pi \left(\sqrt{\frac{S}{6\pi}} \right) < 0$$

\therefore Volume is maximum when

$$r^2 = \frac{S}{6\pi} \text{ or } S = 6\pi r^2$$

$$\text{From (i) } h = \frac{6\pi r^2 - 2\pi r^2}{2\pi}$$

$$\Rightarrow h = 2r$$

Hence volume is maximum when height is twice the radius, i.e., height is equal to the diameter of base.

50. Let ABC be a right angled triangle with $BC = x$, $AC = y$ such that $x + y = k$, where k is any constant.

Let θ be the angle between the base and the hypotenuse.

Let P be the area of the triangle.

$$P = \frac{1}{2} \times BC \times AB = \frac{1}{2} \times x \sqrt{y^2 - x^2} \Rightarrow P^2 = \frac{x^2}{4} (y^2 - x^2)$$

$$\Rightarrow P^2 = \frac{x^2}{4} [(k-x)^2 - x^2]$$

$$\Rightarrow P^2 = \frac{k^2 x^2 - 2kx^3}{4}$$

$$\text{Let } Q = P^2 \text{ i.e. } Q = \frac{k^2 x^2 - 2kx^3}{4}$$

\therefore P is maximum when Q is maximum.

Differentiating Q w.r.t. x, we get

$$\frac{dQ}{dx} = \frac{2k^2 x - 6kx^2}{4} \quad \dots(i)$$

For maximum or minimum area,

$$\frac{dQ}{dx} = 0 \Rightarrow k^2 x - 3kx^2 = 0 \Rightarrow x = \frac{k}{3}$$

Differentiating (i) w.r.t. x, we get

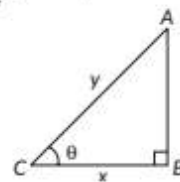
$$\frac{d^2Q}{dx^2} = \frac{2k^2 - 12kx}{4}$$

$$\therefore \left[\frac{d^2Q}{dx^2} \right]_{x=\frac{k}{3}} = \frac{-k^2}{2} < 0$$

Thus, Q is maximum when $x = \frac{k}{3}$

\Rightarrow P is maximum at $x = \frac{k}{3}$

$$\therefore \cos \theta = \frac{x}{y} = \frac{k/3}{2k/3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$



Answer Tips

\Rightarrow Remember $\cos \theta = \frac{\text{Base}}{\text{Hypotenuse}}$

So, the area of $\triangle ABC$ is maximum when angle between the hypotenuse and base is $\frac{\pi}{3}$.

Concept Applied

\Rightarrow Area of triangle = $\frac{1}{2} \times \text{Base} \times \text{Height}$

51. Let V and S be the volume and the surface area of a closed cuboid of length = x units, breadth = x units and height = y units respectively.

$$\text{Then, } V = x^2 y \Rightarrow y = \frac{V}{x^2} \quad \dots(i)$$

$$\text{and } S = 2(x^2 + xy + xy) = 2x^2 + 4xy \quad \dots(ii)$$

$$\Rightarrow S = 2x^2 + 4x \left(\frac{V}{x^2} \right) \quad [\text{From (i)}]$$

$$\Rightarrow S = 2x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 4x - \frac{4V}{x^2}$$

For maximum or minimum of S , $\frac{dS}{dx} = 0$

$$\therefore \frac{dS}{dx} = 0 \Rightarrow 4x - \frac{4V}{x^2} = 0 \Rightarrow V = x^3$$

$$\Rightarrow x^2 y = x^3$$

$$\Rightarrow x = y$$

Differentiating (iii) with respect to x , we get

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} = 4 + \frac{8x^2 y}{x^3} = 4 + \frac{8y}{x}$$

$$\Rightarrow \left(\frac{d^2S}{dx^2}\right)_{y=x} = 12 > 0.$$

Thus, S is minimum when $x = y$.

52. Let ABC be a cone of maximum volume inscribed in the sphere.

Let $OD = x$

$$\therefore BD = \sqrt{r^2 - x^2}$$

and $AD = AO + OD$

$$= r + x = \text{altitude of cone.}$$

Let V be the volume of cone.

$$V = \frac{1}{3}\pi(BD)^2(AD) = \frac{1}{3}\pi(r^2 - x^2)(r + x)$$

$$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi[(r^2 - x^2) + (r + x)(-2x)] = \frac{\pi}{3}[r^2 - 3x^2 - 2rx]$$

$$\text{and } \frac{d^2V}{dx^2} = \frac{\pi}{3}[-6x - 2r]$$

For maximum or minimum value $\frac{dV}{dx} = 0$

$$\Rightarrow r^2 - 3x^2 - 2rx = 0$$

$$\Rightarrow r^2 - 3rx + rx - 3x^2 = 0$$

$$\Rightarrow (r - 3x)(r + x) = 0$$

$$\Rightarrow r = 3x$$

$$\Rightarrow x = \frac{r}{3}$$

$$\text{Also, } \left(\frac{d^2V}{dx^2}\right)_{x=\frac{r}{3}} = \frac{\pi}{3}\left[-6\left(\frac{r}{3}\right) - 2r\right]$$

$$= \frac{\pi}{3}[-2r - 2r] = \frac{-4}{3}r\pi < 0$$

$$\Rightarrow V \text{ is maximum when } x = \frac{r}{3}$$

$$\text{and altitude of cone} = AD = r + x = r + \frac{r}{3} = \frac{4r}{3}$$

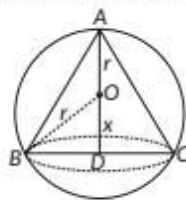
Also, maximum volume of cone when $x = \frac{r}{3}$

$$= \frac{1}{3}\pi\left(r^2 - \frac{r^2}{9}\right)\left(r + \frac{r}{3}\right) = \frac{\pi}{3}\left(\frac{8}{9}r^2\right)\left(\frac{4}{3}r\right)$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi r^3\right) = \frac{8}{27} \text{ (Volume of sphere) cube units.}$$

...(iii)

$$[\because V = x^2 y]$$



$$[\because r + x \neq 0]$$

53. Let $\triangle ABC$ be the given triangle and AD is the altitude of the isosceles triangle ABC .

Since, ' r ' be the radius of the inscribed circle.

So, $OD = OE = OF = r$, where O is the centre of the inscribed circle.

AB and AC are the equal sides.

$$BD = DC$$

$$BD = BE \text{ and } CD = CF$$

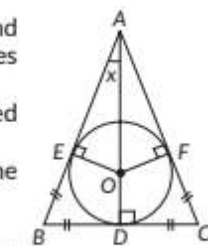
$$\text{From (i) and (ii), } BD = BE = DC = CF$$

$$\text{Similarly, } AE = AF$$

$$\text{Perimeter of the triangle } ABC = AB + BC + AC$$

$$= AE + BE + BD + DC + CF + AF$$

$$= 2AE + 4BD$$



...(i)

...(ii)

...(iii)

...(iv)

(Using (iii) and (iv))

$$\text{In right triangle } OEA, AE = \frac{OE}{\tan x} = \frac{r}{\tan x} \text{ and } AO = \frac{r}{\sin x}$$

$$\text{In right triangle } ABD, BD = AD \tan x$$

$$= (AO + OD) \tan x = \left(\frac{r}{\sin x} + r\right) \tan x$$

Let P be the perimeter of a triangle ABC .

$$\text{So perimeter, } (P) = 2AE + 4BD$$

$$= \frac{2r}{\tan x} + 4\left(\frac{r}{\sin x} + r\right) \tan x$$

$$\Rightarrow P(x) = r(2\cot x + 4\sec x + 4\tan x)$$

For maximum or minimum perimeter, $\frac{dP(x)}{dx} = 0$

$$\Rightarrow \frac{dP(x)}{dx} = r(-2\operatorname{cosec}^2 x + 4\sec x \tan x + 4\sec^2 x) = 0$$

$$\Rightarrow r\left(-\frac{2}{\sin^2 x} + \frac{4\sin x}{\cos^2 x} + \frac{4}{\cos^2 x}\right) = 0$$

$$\Rightarrow r\left(\frac{-2\cos^2 x + 4\sin^3 x + 4\sin^2 x}{\sin^2 x \cos^2 x}\right) = 0$$

$$\Rightarrow -2(1 - \sin^2 x) + 4\sin^3 x + 4\sin^2 x = 0$$

$$\Rightarrow 2\sin^3 x + 3\sin^2 x - 1 = 0$$

$$\Rightarrow (\sin x + 1)(2\sin^2 x + \sin x - 1) = 0$$

$\sin x$ cannot be -1 because ' x ' cannot be more than 90° .

$$\text{So, } 2\sin^2 x + \sin x - 1 = 0$$

$$\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$$

Again, $\sin x$ cannot be -1 .

$$\text{So } 2\sin x - 1 = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = 30^\circ$$

$$\frac{d^2P(x)}{dx^2} = r[4\operatorname{cosec}^2 x \cot x + 4\sec x \tan^2 x + 4\sec^3 x + 8\sec^2 x \tan x]$$

$$\therefore \left[\frac{d^2P}{dx^2}\right]_{x=30^\circ} > 0$$

So it is a point of minima for $P(x)$

$$\text{Hence, least perimeter} = [P(x)]_{x=30^\circ} = r(2\cot 30^\circ + 4\sec 30^\circ + 4\tan 30^\circ)$$

$$= r\left(2\sqrt{3} + 4 \times \frac{2}{\sqrt{3}} + 4 \times \frac{1}{\sqrt{3}}\right) = r\left(\frac{18}{\sqrt{3}}\right) = 6\sqrt{3}r$$

Key Points

➤ If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

54. Surface area of cuboid = $2(lb + bh + hl)$
 $= 2\left(2x^2 + \frac{2x^2}{3} + \frac{x^2}{3}\right) = 6x^2$

Let radius of the sphere be r .

Surface area of sphere = $4\pi r^2$

Therefore, $6x^2 + 4\pi r^2 = k$ (constant)

Now, sum of volumes of cuboid and sphere is

$$V = \frac{2}{3}x^3 + \frac{4}{3}\pi r^3$$

Putting the value of r from (i) into (ii), we get

$$V = \frac{2}{3}x^3 + \frac{4}{3}\pi \left(\frac{k-6x^2}{4\pi}\right)^{3/2}$$

Differentiating (iii) w.r.t. 'x', we get

$$\frac{dV}{dx} = 2x^2 + \frac{4}{3}\pi \left(\frac{1}{4\pi}\right)^{3/2} \cdot \frac{3}{2}(k-6x^2)^{1/2}(-12x)$$

For minimum or maximum value, $\frac{dV}{dx} = 0$

$$\Rightarrow \frac{dV}{dx} = 2x^2 + \frac{4}{3}\pi \left(\frac{1}{4\pi}\right)^{3/2} \cdot \frac{3}{2}(k-6x^2)^{1/2}(-12x) = 0$$

$$\Rightarrow 2x^2 = \left(\frac{1}{4\pi}\right)^{1/2} (k-6x^2)^{1/2}(6x)$$

$$\Rightarrow 2x^2 = \left(\frac{1}{4\pi}\right)^{1/2} (4\pi r^2)^{1/2}(6x)$$

$$\Rightarrow x = 3r$$

Differentiating (iv) w.r.t. 'x', we get

$$\frac{d^2V}{dx^2} = 4x - \left(\frac{1}{4\pi}\right)^{1/2} \left[(6)(k-6x^2)^{1/2} + (6x) \frac{(-12x)}{2(k-6x^2)^{1/2}} \right]$$

$$\text{Now, } \left[\frac{d^2V}{dx^2}\right]_{x=3r} = \frac{24\pi r^2 + 324r^2}{4\pi r} > 0$$

Thus, V is minimum at $x = 3r$.

Further, minimum value of sum of their volume

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi r^3 = \frac{2}{3}x^3 + \frac{4}{3}\pi \left(\frac{x}{3}\right)^3 \quad [\because r = \frac{x}{3}]$$

$$= \frac{2}{3}x^3 + \frac{4}{3}\pi \frac{x^3}{27} = \frac{2}{3}x^3 \left(1 + \frac{2\pi}{27}\right) = \frac{2}{3}x^3 \left(1 + \frac{44}{189}\right)$$

$$= \frac{2}{3}x^3 \cdot \frac{233}{189} = \frac{466}{567}x^3$$

55. We have, $f(x) = \sin x - \cos x$

$$\Rightarrow f'(x) = \cos x + \sin x$$

For maxima or minima, $f'(x) = 0$

$$\Rightarrow \cos x + \sin x = 0 \Rightarrow \tan x = -1$$

$$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$f''(x) = -\sin x + \cos x$$

$$\text{At } x = \frac{3\pi}{4}, f''(x) = -\sin \frac{3\pi}{4} + \cos \frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\begin{aligned} \text{At } x = \frac{7\pi}{4}, f''(x) &= -\sin \frac{7\pi}{4} + \cos \frac{7\pi}{4} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2} \end{aligned}$$

Since $f''(x) < 0$ when $x = \frac{3\pi}{4}$

$\therefore f(x)$ has local maxima at $x = \frac{3\pi}{4}$

...(i) Since $f''(x) > 0$ when $x = \frac{7\pi}{4}$

...(ii) $\therefore f(x)$ has local minima at $x = \frac{7\pi}{4}$

\therefore Local maximum value at $x = \frac{3\pi}{4}$ is

$$f(x) = \sin \frac{3\pi}{4} - \cos \frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

...(iii) Local minimum value at $x = \frac{7\pi}{4}$ is

$$f(x) = \sin \frac{7\pi}{4} - \cos \frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

Concept Applied

➤ If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima.

➤ If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

[From (i)]

56. Let $P(h, k)$ be the coordinates of the point on given parabola.

$$\therefore k = h^2 + 7h + 2 \quad \dots(i)$$

The distance S of P from the straight line $-3x + y + 3 = 0$ is

$$S = \frac{|-3h + k + 3|}{\sqrt{10}} = \frac{|-3h + h^2 + 7h + 2 + 3|}{\sqrt{10}} \quad \text{[From (i)]}$$

$$= \frac{|h^2 + 4h + 5|}{\sqrt{10}} \therefore S = \frac{f(h)}{\sqrt{10}}$$

$\Rightarrow S$ will be maximum or minimum according as $f(h)$ is maximum or minimum.

$$\text{Since, } f(h) = h^2 + 4h + 5$$

$$f'(h) = 2h + 4$$

For maxima or minima, $f'(h) = 0 \Rightarrow 2h + 4 = 0 \Rightarrow h = -2$

Also, $f''(h) = 2 > 0$ when $h = -2$

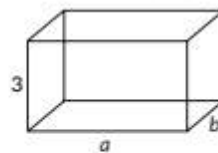
S is minimum at $h = -2$

Putting this value in (i), we get

$$k = (-2)^2 + 7(-2) + 2 = 4 - 14 + 2 = -8$$

\therefore The required coordinates are $(-2, -8)$

57. Let a m and b m be the sides of the base of the tank.



\therefore Volume of the tank = $a \cdot b \cdot 3 = 75 \text{ m}^3$ (given)

$$\Rightarrow ab = 25 \Rightarrow b = \frac{25}{a}$$

...(i)

If C is the total cost in rupees, then
 $C = a \times b \times 100 + 2 \times 3 \times a \times 50 + 2 \times 3 \times b \times 50$
 $= 100ab + 300(a + b) = 100 \times 25 + 300\left(a + \frac{25}{a}\right)$

$$\Rightarrow C = 2500 + 300\left(a + \frac{25}{a}\right)$$

Differentiating w.r.t. a , we get

$$\frac{dC}{da} = 300\left(1 - \frac{25}{a^2}\right) \text{ and}$$

$$\frac{d^2C}{da^2} = 300\left(0 + \frac{25 \times 2}{a^3}\right) = \frac{300 \times 50}{a^3}$$

For maximum or minimum cost,

$$\frac{dC}{da} = 0 \Rightarrow 1 - \frac{25}{a^2} = 0 \Rightarrow a = 5 \text{ m}$$

and from (i) $b = 5 \text{ m}$

At $a = 5$; $\frac{d^2C}{da^2} > 0 \Rightarrow C$ is minimum.

Hence, the least cost of the tank is

$$C = \left[2500 + 300\left(5 + \frac{25}{5}\right)\right] = [2500 + 3000] = ₹5500.$$

Concept Applied

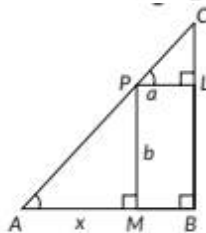
Volume of Cuboid = $l \times b \times h$

58. Let P be any point on the hypotenuse of the given right triangle.

Let $PL = a$, $PM = b$
 and $AM = x$.

Clearly, $\triangle CPL$ and $\triangle PAM$ are similar

$$\therefore \frac{PL}{CL} = \frac{AM}{PM} \Rightarrow CL = \frac{PL \cdot PM}{AM} = \frac{a \cdot b}{x}$$



Now $AB = x + a$ and $BC = b + CL = b + \frac{ab}{x}$.

From right $\triangle ABC$, $AC^2 = AB^2 + BC^2$

Taking $l = AC^2$

$$\therefore l = (x+a)^2 + \left(b + \frac{ab}{x}\right)^2 = (x+a)^2 + b^2 \left(1 + \frac{a}{x}\right)^2$$

Differentiating w.r.t. x , we get

$$\frac{dl}{dx} = 2(x+a) + b^2 \cdot 2 \left(1 + \frac{a}{x}\right) \cdot \frac{-a}{x^2}$$

$$= 2(x+a) - \frac{2ab^2(x+a)}{x^3} = 2(x+a) \left[1 - \frac{ab^2}{x^3}\right]$$

$$\text{and } \frac{d^2l}{dx^2} = 2 \cdot 1 \left[1 - \frac{ab^2}{x^3}\right] + 2(x+a) \cdot \frac{3ab^2}{x^4}$$

\therefore For maximum and minimum value of l ,

$$\frac{dl}{dx} = 0 \Rightarrow x+a=0 \text{ or } 1 - \frac{ab^2}{x^3} = 0$$

As $x = AM \neq 0 \therefore$ Reject $x+a=0$

$$\therefore x^3 = ab^2 \Rightarrow x = a^{1/3}b^{2/3}$$

For this value of x , clearly $\frac{d^2l}{dx^2} > 0$

$\therefore l$ and consequently the hypotenuse AC is minimum (least).

Hence, the least value of AC is given by

$$AC = \sqrt{(x+a)^2 + b^2 \left(1 + \frac{a}{x}\right)^2} \text{ where } x = a^{1/3}b^{2/3}$$

$$= \sqrt{(x+a)^2 + \frac{b^2}{x^2}(x+a)^2} = (x+a) \sqrt{1 + \frac{b^2}{x^2}}$$

$$= \left(\frac{x+a}{x}\right) \sqrt{b^2 + x^2} = \left(\frac{a^{1/3}b^{2/3} + a}{a^{1/3}b^{2/3}}\right) \sqrt{b^2 + a^{2/3}b^{4/3}}$$

$$= \frac{a^{1/3}(b^{2/3} + a^{2/3})}{a^{1/3}b^{2/3}} \cdot b^{2/3} \sqrt{b^{2/3} + a^{2/3}} = (a^{2/3} + b^{2/3})^{3/2}$$

59. Let r and h be the radius and height of the cylindrical can respectively.

Therefore, the total surface area of the closed cylinder is given by

$$S = 2\pi rh + 2\pi r^2 = 2\pi r(r+h) \quad \dots(i)$$

Given volume of the can = $128\pi \text{ cm}^3$

$$\text{Also volume (V)} = \pi r^2 h \quad \dots(ii)$$

$$\Rightarrow \pi r^2 h = 128\pi \Rightarrow h = \frac{128}{r^2} \quad \dots(iii)$$

Putting the value of h in equation (i), we get

$$S = 2\pi r \left(r + \frac{128}{r^2}\right) = 2\pi r^2 + \frac{256}{r} \pi \quad \dots(iv)$$

Differentiating (iv) w.r.t. r , we get

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} \quad \dots(v)$$

Substituting $\frac{dS}{dr} = 0$ for critical points, we get

$$4\pi r - \frac{256\pi}{r^2} = 0 \Rightarrow r^3 = 64 \Rightarrow r = 4 \text{ cm}$$

Differentiating (v) w.r.t. r , we get

$$\frac{d^2S}{dr^2} = 4\pi - 256\pi(-2r^{-3}) = 4\pi + \frac{512}{r^3} \pi \quad \therefore \left[\frac{d^2S}{dr^2}\right]_{r=4} > 0$$

Thus the total surface area of the cylinder is minimum when $r = 4$.

$$\text{From (iii), we have } h = \frac{128}{r^2} = \frac{128}{16} = 8$$

Thus radius = 4 cm and height = 8 cm.

Commonly Made Mistake

Remember the difference between first derivative test and second derivative test for finding local maxima and local minima.

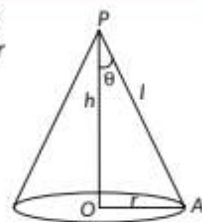
60. Let θ be the semi-vertical angle of the cone, V its volume, h its height, r base radius and slant height l .

Then from $\triangle OAP$,

$$r = l \sin \theta, h = l \cos \theta$$

$$\text{Now, } V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi l^2 \sin^2 \theta \cdot l \cos \theta$$

$$= \frac{1}{3} \pi l^3 \sin^2 \theta \cdot \cos \theta$$



$$\Rightarrow \frac{dV}{d\theta} = \frac{1}{3}\pi l^3 (2\sin\theta \cdot \cos\theta \cdot \cos\theta - \sin^2\theta \cdot \sin\theta)$$

$$= \frac{1}{3}\pi l^3 \sin\theta (2\cos^2\theta - \sin^2\theta)$$

$$\text{and } \frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3 [\cos\theta (2\cos^2\theta - \sin^2\theta) + \sin\theta (-4\cos\theta \sin\theta - 2\sin\theta \cos\theta)]$$

$$= \frac{1}{3}\pi l^3 [\cos\theta (2\cos^2\theta - \sin^2\theta) - 6\sin^2\theta \cos\theta]$$

For maximum or minimum value of V ,

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta (2\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow \sin\theta = 0 \text{ or } 2\cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow 2\cos^2\theta - (1 - \cos^2\theta) = 0 \quad [\text{Note: } \sin\theta \neq 0 \text{ as } \theta \neq 0]$$

$$\Rightarrow \cos^2\theta = \frac{1}{3} \Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\text{For } \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3 \left[\frac{1}{\sqrt{3}} \left(2 \cdot \frac{1}{3} - \frac{2}{3} \right) - 6 \cdot \frac{2}{3} \cdot \frac{1}{\sqrt{3}} \right]$$

$$= \frac{1}{3}\pi l^3 \left(-\frac{4}{\sqrt{3}} \right) < 0$$

$$\therefore V \text{ is maximum for } \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

61. Let r, h, l, V and S be respectively the base radius, height, slant height, volume and curved surface area of the cone. Then,

$$l^2 = r^2 + h^2,$$

$$V = \frac{1}{3}\pi r^2 h \quad \dots (i)$$

$$\text{and } S = \pi r l = \pi r \sqrt{r^2 + h^2}$$

$$\Rightarrow S^2 = \pi^2 r^2 (r^2 + h^2)$$

$$= \pi^2 \frac{3V}{\pi h} \left(\frac{3V}{\pi h} + h^2 \right)$$

$$= 3\pi V \left(\frac{3V}{\pi h^2} + h \right)$$

For S to be least, S^2 is also least.

$$\therefore \frac{dS^2}{dh} = 3\pi V \left(\frac{-6V}{\pi h^3} + 1 \right) \text{ and}$$

$$\frac{d^2S^2}{dh^2} = 3\pi V \left(\frac{-6V}{\pi} \right) \cdot \frac{-3}{h^4} = \frac{54V^2}{h^4}$$

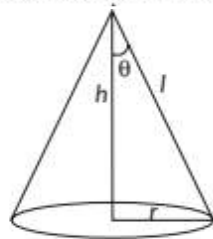
For maximum or minimum S (and so S^2),

$$\frac{dS^2}{dh} = 0 \Rightarrow 6V = \pi h^3$$

$$\Rightarrow h = \left(\frac{6V}{\pi} \right)^{1/3}$$

$$\text{For this value of } h, \frac{d^2S^2}{dh^2} = \frac{54V^2}{h^4} > 0$$

$\Rightarrow S^2$ and therefore S is least.



[Using (i)]

$$\cot\theta = \frac{h}{r} = \frac{h}{\sqrt{3V/\pi h}} = \sqrt{\frac{\pi}{3V}} \cdot h^{3/2} \quad [\text{From (i)}]$$

$$\Rightarrow \cot\theta = \sqrt{\frac{\pi}{3V}} \sqrt{\frac{6V}{\pi}} = \sqrt{2} \quad [\text{From (ii)}]$$

\Rightarrow The semi vertical angle, $\theta = \cot^{-1}\sqrt{2}$

Answer Tips

$$\Rightarrow \cot\theta = \frac{\text{Base}}{\text{Perpendicular}}$$

62. Let a be the side of the given square and r be the radius of the circle.

By hypothesis

$$4a + 2\pi r = k$$

$$\Rightarrow a = \frac{k - 2\pi r}{4} \quad \dots (i)$$

Let A = Sum of areas of the circle and the square

$$\Rightarrow A = \pi r^2 + a^2 = \pi r^2 + \frac{1}{16}(k - 2\pi r)^2 \quad [\text{Using (i)}]$$

$$\Rightarrow \frac{dA}{dr} = 2\pi r + \frac{1}{16} \cdot 2(k - 2\pi r) \cdot (-2\pi)$$

$$= 2\pi r - \frac{\pi}{4}(k - 2\pi r)$$

$$\text{and } \frac{d^2A}{dr^2} = 2\pi - \frac{\pi}{4}(0 - 2\pi) = 2\pi + \frac{\pi^2}{2}$$

For maxima or minima,

$$\frac{dA}{dr} = 0 \Rightarrow 2\pi r - \frac{\pi}{4}(k - 2\pi r) = 0$$

$$\Rightarrow 8r - k + 2\pi r = 0 \Rightarrow (8 + 2\pi)r = k \Rightarrow r = \frac{k}{2\pi + 8}$$

$$\text{For this value of } r, \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

$$\therefore A \text{ is minimum (least), when } r = \frac{k}{2\pi + 8}$$

$$\text{From (i), } a = \frac{k - 2\pi \cdot \frac{k}{2\pi + 8}}{4}$$

$$= \frac{k}{4} \left(\frac{2\pi + 8 - 2\pi}{2\pi + 8} \right) = \frac{2k}{2\pi + 8} = 2r$$

\therefore Area is least, when $a = 2r$.

63. Let r and h be the base radius and height of the cylinder respectively and volume of cylinder, $V = \pi r^2 h$

$$\Rightarrow h = \frac{V}{\pi r^2} \quad \dots (i)$$

Total surface area of the cylinder, $S = 2\pi r h + \pi r^2$

$$\Rightarrow S = 2\pi r \left(\frac{V}{\pi r^2} \right) + \pi r^2 \quad [\text{By using (i)}]$$

$$\Rightarrow S = \frac{2V}{r} + \pi r^2$$

On differentiating w.r.t. r both sides, $\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$

Again differentiating w.r.t. r both sides, $\frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi$

For maxima or minima, $\frac{dS}{dr} = 0$

$$\Rightarrow -\frac{2V}{r^2} + 2\pi r = 0 \Rightarrow 2\pi r = \frac{2V}{r^2}$$

$$\Rightarrow \pi r^3 = V \Rightarrow r = \left(\frac{V}{\pi}\right)^{1/3}$$

$$\therefore \left[\frac{d^2V}{dh^2}\right]_{r=\left(\frac{V}{\pi}\right)^{1/3}} = 4V\left(\frac{\pi}{V}\right) + 2\pi = 6\pi > 0$$

So, S is minimum at $r = \left(\frac{V}{\pi}\right)^{1/3}$

Now, $\pi r^3 = V \Rightarrow \pi r^3 = \pi r^2 h \Rightarrow r = h$

Hence, the cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base.

64. Let $ABCD$ be a rectangle and let the semi-circle be described on the side AB as its diameter.

Let $AB = 2x$ and $AD = 2y$. Let $P = 10$ m be the given perimeter of window.

Therefore, $10 = 2x + 4y + \pi x$

$$\Rightarrow 4y = 10 - 2x - \pi x \quad \dots(i)$$

Area of the window,

$$A = (2x)(2y) + \frac{1}{2}\pi x^2$$

$$\Rightarrow A = 4xy + \frac{1}{2}\pi x^2$$

$$\Rightarrow A = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

$$\Rightarrow A = 10x - 2x^2 - \frac{1}{2}\pi x^2$$

On differentiating w.r.t. x , $\frac{dA}{dx} = 10 - 4x - \pi x$

Again differentiating w.r.t. x , $\frac{d^2A}{dx^2} = -(4 + \pi)$

For maxima or minima,

$$\frac{dA}{dx} = 0 \Rightarrow 10 - 4x - \pi x = 0 \Rightarrow x = \frac{10}{4 + \pi}$$

$$\left[\frac{d^2A}{dx^2}\right]_{x=\frac{10}{4+\pi}} = -(4 + \pi) < 0$$

So, A is maximum at $x = \left(\frac{10}{4 + \pi}\right)$ m.

Now, length of the window is $2x = \left(\frac{20}{4 + \pi}\right)$ m and width is

$$2y = \left(\frac{10}{4 + \pi}\right) \text{ m.}$$

Concept Applied

Area of rectangle = length \times breadth

$$\text{Area of semicircle} = \frac{1}{2}\pi r^2$$

65. Here BA is a diameter of the given circle, of radius $= r$.

Let $\angle CAB = \theta$

Also $\angle ACB = \frac{\pi}{2}$

Now $AC = AB \cos \theta = 2r \cos \theta$

$BC = AB \sin \theta = 2r \sin \theta$

Let Area of $\triangle ABC = \frac{1}{2} \cdot AC \cdot BC$

$$= \frac{1}{2} \cdot 2r \cos \theta \cdot 2r \sin \theta = r^2 \cdot \sin 2\theta$$

$$\Rightarrow \frac{d\Delta}{d\theta} = r^2 \cdot 2 \cos 2\theta \text{ and } \frac{d^2\Delta}{d\theta^2} = -r^2 \cdot 4 \sin 2\theta$$

For maxima or minima,

$$\frac{d\Delta}{d\theta} = 0 \Rightarrow \cos 2\theta = 0$$

$$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$\text{and } \left[\frac{d^2\Delta}{d\theta^2}\right]_{\theta=\frac{\pi}{4}} = -4r^2 \sin \frac{\pi}{2} = -4r^2 < 0$$

Hence, area of $\triangle ABC$ is maximum when

$$\angle CAB = \theta = \frac{\pi}{4} = \angle ABC$$

$$\left[\because \angle ACB = \frac{\pi}{2} \right]$$

$\Rightarrow \triangle ABC$ is isosceles.

Answer Tips

Chain rule of derivative: $\frac{dt}{dx} = \frac{dt}{du} \cdot \frac{du}{dx}$

66. The given parabola is

$$y^2 = 4ax \quad \dots(i)$$

Let $Q(11a, 0)$.

Any point on (i) is $P(at^2, 2at)$

$$\therefore PQ^2 = (at^2 - 11a)^2 + (2at - 0)^2$$

$$\text{Let } l = PQ^2 = a^2 t^4 - 18a^2 t^2 + 121a^2$$

$$\Rightarrow \frac{dl}{dt} = 4a^2 t^3 - 36a^2 t \text{ and } \frac{d^2l}{dt^2} = 12a^2 t^2 - 36a^2$$

For maximum or minimum value of l ,

$$\frac{dl}{dt} = 0 \Rightarrow 4a^2 t(t^2 - 9) = 0 \Rightarrow t = 0, 3, -3$$

$$\text{For } t = 0, \frac{d^2l}{dt^2} = -36a^2 < 0$$

This corresponds to a maximum value of l .

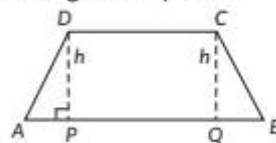
Both for $t = 3$ and -3 ,

$$\frac{d^2l}{dt^2} = 12a^2 \cdot 9 - 36a^2 = 72a^2 > 0$$

\therefore This corresponds to a minimum value of l i.e., of PQ^2 and therefore of PQ .

Thus, there are two such points P with coordinates, $(9a, 6a)$ and $(9a, -6a)$ nearest to the given point Q .

67. Let $ABCD$ be the given trapezium.



Then $AD = DC = CB = 10$ cm

In $\triangle APD$ and $\triangle BQC$

$DP = CQ = h$

$AD = BC = 10$ cm

$\angle DPA = \angle CQB = 90^\circ$

$\therefore \triangle APD \cong \triangle BQC$ (by R.H.S. congruency)

$\Rightarrow AP = QB = x$ cm (Say)

$\therefore AB = AP + PQ + QB$
 $= x + 10 + x = (2x + 10)$ cm

Also from $\triangle APD$, $AP^2 + PD^2 = AD^2$

$\Rightarrow x^2 + h^2 = 10^2$

$\Rightarrow h = \sqrt{100 - x^2}$... (i)

Now, area A of this trapezium is given by

$$A = \frac{1}{2}(AB + DC) \cdot h = \frac{1}{2}(2x + 10 + 10)h$$

$$= (x + 10) \cdot \sqrt{100 - x^2} \quad \dots \text{(ii) [Using (i)]}$$

Differentiating w.r.t. x , we get

$$\frac{dA}{dx} = 1 \cdot \sqrt{100 - x^2} + (x + 10) \cdot \frac{1}{2\sqrt{100 - x^2}} \cdot (-2x)$$

$$= \frac{100 - x^2 - x^2 - 10x}{\sqrt{100 - x^2}} = \frac{-2(x^2 + 5x - 50)}{\sqrt{100 - x^2}}$$

$$= \frac{-2(x + 10)(x - 5)}{\sqrt{100 - x^2}}$$

For maximum or minimum value of A , $\frac{dA}{dx} = 0$

$$\Rightarrow (x + 10)(x - 5) = 0$$

$$\Rightarrow x = 5 \quad (\text{Reject } x = -10 \text{ as } x \neq 0)$$

For this value of x , $\frac{dA}{dx}$ changes sign from positive to negative.

$\therefore A$ is maximum at $x = 5$.

$$\text{From (ii), the maximum value of } A = (5 + 10) \cdot \sqrt{100 - 5^2}$$

$$= 15\sqrt{75} = 75\sqrt{3} \text{ sq.cm.}$$

Key Points

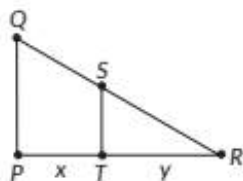
Area of Trapezium = $\frac{1}{2}(a + b)h$, where a and b are parallel side and h is the height of the trapezium.

CBSE Sample Questions

1. Let PQ represent the height of the street light from the ground. At any time t seconds, let the man represent as ST of height 1.6 m be at a distance of x m from PQ and the length of his shadow TR be y m.

Using similarity of triangles, we have $\frac{4}{1.6} = \frac{x + y}{y}$ (1/2)

$$\Rightarrow 3y = 2x$$



Differentiating both sides w.r.t. t , we get $3 \frac{dy}{dt} = 2 \frac{dx}{dt}$

$$\Rightarrow \frac{dy}{dt} = \frac{2}{3} \times 0.3 \Rightarrow \frac{dy}{dt} = 0.2 \quad (1/2)$$

At any time t seconds, the tip of his shadow is at a distance of $(x + y)$ m from PQ .

The rate at which the tip of his shadow moving

$$= \left(\frac{dx}{dt} + \frac{dy}{dt} \right) \text{ m/s} = 0.5 \text{ m/s} \quad (1/2)$$

The rate at which his shadow is lengthening

$$= \frac{dy}{dt} \text{ m/s} = 0.2 \text{ m/s} \quad (1/2)$$

2. (b): We have, $f(x) = x^2 - 4x + 6$

$$\Rightarrow f'(x) = 2x - 4$$

$\therefore f(x)$ is strictly increasing.

$$\therefore f'(x) > 0$$

$$\Rightarrow 2x - 4 > 0 \Rightarrow x > 2$$

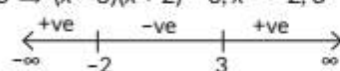
$$\Rightarrow x \in (2, \infty)$$
 (1)

3. (b): We have, $f(x) = 2x^3 - 3x^2 - 36x + 7$

$$\Rightarrow f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$$

$$= 6(x - 3)(x + 2)$$

$$f'(x) = 0 \Rightarrow (x - 3)(x + 2) = 0, x = -2, 3$$



$\therefore f(x)$ is strictly increasing in $(-\infty, -2) \cup (3, \infty)$ and strictly decreasing in $(-2, 3)$. (1)

4. (b): We have, $f(x) = x + \cos x + b$

$$\Rightarrow f'(x) = 1 - \sin x \Rightarrow f'(x) \geq 0 \forall x \in \mathbb{R}$$

\Rightarrow No such value of b exists (1)

5. We have, $f(x) = \tan x - 4x$

$$\Rightarrow f'(x) = \sec^2 x - 4 \quad (1)$$

(a) For $f(x)$ to be strictly increasing, $f'(x) > 0$

$$\Rightarrow \sec^2 x - 4 > 0 \Rightarrow \sec^2 x > 4$$

$$\Rightarrow \cos^2 x < \frac{1}{4} \Rightarrow \cos^2 x < \left(\frac{1}{2}\right)^2$$

$$\Rightarrow -\frac{1}{2} < \cos x < \frac{1}{2} \Rightarrow \frac{\pi}{3} < x < \frac{\pi}{2} \quad \left(\because x \in \left(0, \frac{\pi}{2}\right) \right) \quad (1)$$

(b) For $f(x)$ to be strictly decreasing, $f'(x) < 0$

$$\Rightarrow \sec^2 x - 4 < 0 \Rightarrow \sec^2 x < 4$$

$$\Rightarrow \cos^2 x > \frac{1}{4} \Rightarrow \cos^2 x > \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \cos x > \frac{1}{2} \quad \left[\because x \in \left(0, \frac{\pi}{2}\right) \right] \Rightarrow 0 < x < \frac{\pi}{3} \quad (1)$$

6. (c): We have, $f(x) = 2\cos x + x$

$$\Rightarrow f'(x) = -2\sin x + 1$$

$$\Rightarrow f''(x) = -2\cos x$$

For critical points, $f'(x) = 0$

$$\Rightarrow -2\sin x + 1 = 0$$

$$\Rightarrow \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) \quad \left(\because x \in \left[0, \frac{\pi}{2}\right] \right)$$

$$\Rightarrow x = \frac{\pi}{6}$$

$$f''(x) \left(\text{at } x = \frac{\pi}{6} \right) = -2 \cos \frac{\pi}{6} = -\sqrt{3} < 0$$

So, $x = \frac{\pi}{6}$ is the point of maxima.

$$\text{Now, } f(0) = 2 \text{ and } f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 1.57$$

$$\Rightarrow \text{Least value of } f(x) = 1.57 \text{ i.e., } \frac{\pi}{2}$$

$$7. \text{ (c): We have, } f(x) = (10+x)\sqrt{100-x^2}$$

Which will give some real area if $-10 < x < 10$

$$\Rightarrow f'(x) = \frac{(10+x)(-2x)}{2\sqrt{100-x^2}} + \sqrt{100-x^2} \times 1$$

$$\Rightarrow f'(x) = \frac{-2x^2 - 10x + 100}{\sqrt{100-x^2}}$$

For critical points, put $f'(x) = 0$

$$\Rightarrow x^2 + 5x - 50 = 0$$

$$\Rightarrow (x+10)(x-5) = 0$$

$$\Rightarrow x = -10 \text{ or } 5 \Rightarrow x = 5 \quad [\because -10 < x < 10]$$

Now, $f''(x)$

$$= \frac{(\sqrt{100-x^2})(-4x-10) + (2x^2+10x-100) \times \frac{1}{2} \frac{(-2x)}{\sqrt{100-x^2}}}{(100-x^2)}$$

$$= \frac{2x^3 - 300x - 1000}{(100-x^2)^{3/2}} \Rightarrow f''(5) = \frac{-30}{\sqrt{75}} < 0$$

\therefore Maximum area of trapezium

$$= (10+5)(\sqrt{75}) = 75\sqrt{3} \text{ cm}^2$$

$$8. \text{ (c): Let } f(x) = [x(x-1)+1]^{1/3}, 0 \leq x \leq 1$$

$$\Rightarrow f'(x) = \frac{2x-1}{3(x^2-x+1)^{2/3}}$$

For critical points, put $f'(x) = 0$

$$\Rightarrow x = \frac{1}{2} \in [0, 1]$$

$$\text{Now, } f(0) = 1, f\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)^{1/3} \text{ and } f(1) = 1$$

\therefore Maximum value of $f(x)$ is 1. (1)

9. (d): Let F be the fuel cost per hour and v be the speed of train in km/hr.

According to question, we have,

$F \propto v^2 \Rightarrow F = kv^2$, where k is proportionality constant

$$\Rightarrow 48 = k(16)^2 \Rightarrow k = \frac{3}{16} \quad (1)$$

10. (b): Let total cost of running the train be C .

$$\text{Then, } C = \frac{3}{16}v^2t + 1200t$$

$$\text{Now, distance covered} = 500 \text{ km} \Rightarrow \text{Time} = \frac{500}{v} \text{ hrs}$$

\therefore Total cost of running the train for 500 km

$$= \frac{3}{16}v^2\left(\frac{500}{v}\right) + 1200\left(\frac{500}{v}\right)$$

$$\Rightarrow C = \frac{375}{4}v + \frac{600000}{v} \quad (1)$$

$$11. \text{ (c): We have, } \frac{dC}{dv} = \frac{375}{4} - \frac{600000}{v^2}$$

$$\text{Put } \frac{dC}{dv} = 0 \Rightarrow v^2 = \frac{600000 \times 4}{375} = 6400$$

$$\Rightarrow v = 80 \text{ km/h}$$

$$\frac{d^2C}{dv^2} = \frac{2 \times 600000}{v^3} > 0, \text{ for } v = 80$$

\therefore Most economical speed is 80 km/h. (1)

12. (c): Fuel cost for running the train for 500 km

$$= \frac{3}{16}v^2\left(\frac{500}{v}\right)$$

$$= \frac{375}{4}v = \frac{375}{4} \times 80 = ₹ 7500 \quad (1)$$

13. (d): Total cost for running the train for 500 km

$$= \frac{375}{4}v + \frac{600000}{v}$$

$$= \frac{375 \times 80}{4} + \frac{600000}{80} = ₹ 15000 \quad (1)$$

14. (i) (b): We have, perimeter of floor = 200 m

$$\Rightarrow 2x + 2\pi\left(\frac{y}{2}\right) = 200$$

$$\Rightarrow 2x + \pi y = 200 \quad \dots (i) \quad (1)$$

(ii) (a): Area of rectangular region (A) = xy

$$= x\left(\frac{200-2x}{\pi}\right) \quad [\text{Using (i)}]$$

$$= \frac{2}{\pi}(100x - x^2) \quad (1)$$

(iii) (c): We have, $A = \frac{2}{\pi}(100x - x^2)$

$$\Rightarrow \frac{dA}{dx} = \frac{2}{\pi}(100 - 2x)$$

For maximum or minimum, $\frac{dA}{dx} = 0$

$$\Rightarrow 100 - 2x = 0 \Rightarrow x = 50$$

$$\text{Now, } \left[\frac{d^2A}{dx^2}\right]_{x=50} = -\frac{4}{\pi} < 0$$

Thus, A is maximum at $x = 50$.

$$\text{Thus, maximum value of } A = \frac{2}{\pi}(5000 - 2500)$$

$$= \frac{5000}{\pi} \text{ m}^2 \quad (1)$$

(iv) (a): Let P be the area of the whole floor.

$$\text{Then, } P = xy + \pi\left(\frac{y}{2}\right)^2 = xy + \frac{\pi}{4}y^2 = y\left(x + \frac{\pi}{4}y\right)$$

$$= \left(\frac{200-2x}{\pi}\right)\left(\frac{200+2x}{4}\right) \quad [\text{Using (i)}]$$

$$= \frac{40000 - 4x^2}{4\pi} = \frac{10000 - x^2}{\pi}$$

$$\therefore \frac{dP}{dx} = -\frac{2x}{\pi}$$

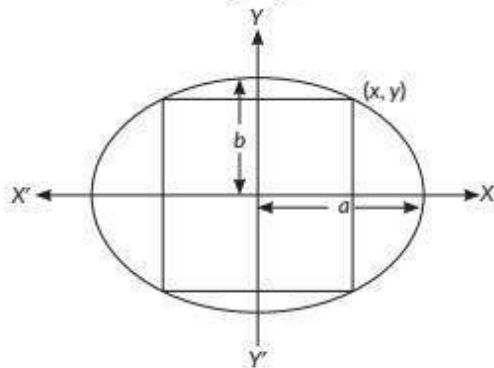
For maximum or minimum, $\frac{dP}{dx} = 0 \Rightarrow x = 0$

$$\text{Now, } \frac{d^2P}{dx^2} = -\frac{2}{\pi} < 0$$

So, P is maximum at $x = 0$ m.

(v) (d)

15. (i) Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let $(x, y) = \left(x, \frac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle.

The area function $A = 2x \times 2 \times \frac{b}{a}\sqrt{a^2 - x^2}$

$$= \frac{4b}{a}x\sqrt{a^2 - x^2}, x \in (0, a)$$

(ii) The first derivative of function is

$$\frac{dA}{dx} = \frac{4b}{a} \left[x \times \frac{-x}{\sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2} \right]$$

$$= \frac{4b}{a} \times \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} = -\frac{4b}{a} \times \frac{2\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right)}{\sqrt{a^2 - x^2}}$$

To find the critical point, put $\frac{dA}{dx} = 0$

$$\Rightarrow x = \frac{a}{\sqrt{2}}$$

So, $x = \frac{a}{\sqrt{2}}$ is the critical point.

(iii) For the values of x less than $\frac{a}{\sqrt{2}}$ and close to

$\frac{a}{\sqrt{2}}$, $\frac{dA}{dx} > 0$ and for the values of x greater than $\frac{a}{\sqrt{2}}$ and

close to $\frac{a}{\sqrt{2}}$, $\frac{dA}{dx} < 0$. (1)

Hence, by the first derivative test, there is a local maximum

(1) at the critical point $x = \frac{a}{\sqrt{2}}$.

(1) Since there is only one critical point, therefore, the area of the soccer field is maximum at this critical point $x = \frac{a}{\sqrt{2}}$. (1)

Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

OR

$$A = 2x \times 2 \times \frac{b}{a}\sqrt{a^2 - x^2}, x \in (0, a)$$

Squaring both sides, we get

$$Z = A^2 = \frac{16b^2}{a^2}x^2(a^2 - x^2) = \frac{16b^2}{a^2}(x^2a^2 - x^4), x \in (0, a) \quad (1/2)$$

(1/2) A is maximum when Z is maximum

To find the critical point, put $\frac{dZ}{dx} = 0$ (1/2)

$$\frac{dZ}{dx} = \frac{16b^2}{a^2}(2xa^2 - 4x^3) = \frac{32b^2}{a^2}x(a + \sqrt{2}x)(a - \sqrt{2}x)$$

(1/2) To find the critical point, put $\frac{dZ}{dx} = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$

The second derivative is: $\frac{d^2Z}{dx^2} = \frac{32b^2}{a^2}(a^2 - 6x^2)$

$$\therefore \left(\frac{d^2Z}{dx^2}\right)_{x=\frac{a}{\sqrt{2}}} = \frac{32b^2}{a^2}(a^2 - 3a^2) = -64b^2 < 0 \quad (1/2)$$

(1/2) Hence, by the second derivative test, there is a local maximum value of Z at the critical point $x = \frac{a}{\sqrt{2}}$. (1/2)

Since there is only one critical point, therefore, Z is maximum at $x = \frac{a}{\sqrt{2}}$, hence, A is maximum at $x = \frac{a}{\sqrt{2}}$.

(1/2) Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.