Dot & Cross Products of Two Vectors

1 Marks Questions

1. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}|=13$ and $|\vec{a}|=5$, then find the value of $|\vec{b}|$. All India 2014

Given, $|\vec{a}+\vec{b}|=13,$ and $|\vec{a}|=5$

Now,

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}$$

$$=|\vec{a}|^2+0+0+|\vec{b}|^2$$

$$[\because \vec{a}\cdot\vec{a}=|\vec{a}|^2, \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}=0 \text{ as } \vec{a}\perp\vec{b}]$$

$$\Rightarrow |\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2$$

$$\Rightarrow (13)^2=(5)^2+|\vec{b}|^2$$

$$\Rightarrow 169=25+|\vec{b}|^2 \Rightarrow 169-25=|\vec{b}|^2$$

$$\Rightarrow 144=|\vec{b}|^2 \Rightarrow |\vec{b}|=12$$ **(1)**

2. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. **Delhi 2014**

Given, $|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{a} + \vec{b}| = 1$

Now, $|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$

$$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2$$

$$[\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \text{ and } \vec{x} \cdot \vec{x} = |\vec{x}|^2]$$

$$\Rightarrow \quad 1 = 1 + 2\,\vec{a} \cdot \vec{b} + 1$$

$$\Rightarrow \quad 2\,\vec{a} \cdot \vec{b} = -1$$

$$\Rightarrow |\vec{a}||\vec{b}| \cos\theta = -\frac{1}{2} \quad [\because \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta]$$

$$\Rightarrow \quad \cos\theta = -\frac{1}{2} \quad [\because |\vec{a}| = |\vec{b}| = 1]$$

$$\Rightarrow \quad \cos\theta = \cos\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3}$$

Hence, angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$. **(1)**

3. Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$. **Delhi 2014**

The projection of vector $\vec{a}$ on vector $\vec{b}$ is given by $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$.

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

The projection of vector $\vec{a}$ on the vector $\vec{b}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{1\times 2 - 3\times 3 + 7\times 6}{\sqrt{(2)^2 + (-3)^2 + (6)^2}}$$

$$= \frac{2 - 9 + 42}{\sqrt{4+9+36}} = \frac{35}{\sqrt{49}} = \frac{35}{7} = 5 \quad \textbf{(1)}$$

4. Write the projection of vector $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$. **Foreign 2014**

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{j}$

The projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{1\times 0 + 1\times 1 + 1\times 0}{\sqrt{1^2}} = 1$$

Hence, the projection of vector $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$ is 1. **(1)**

5. If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3, |\vec{b}| = 2/3$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. **Delhi 2014**

Given, $|\vec{a}| = 3$ and $|\vec{b}| = 2/3$

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Also, given $|\vec{a} \times \vec{b}| = 1$ **(1/2)**

$\Rightarrow \quad |\vec{a}||\vec{b}| \sin\theta = 1 \Rightarrow 3 \times \frac{2}{3} \sin\theta = 1$

$\Rightarrow \quad 2 \sin\theta = 1$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$ **(1/2)**

6. Find $\vec{a} \cdot (\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$.

All India 2014

Given, $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

$$\text{Now, } \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_3 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 2(4-1) - 1(-2-3) + 3(-1-6)$$

$$= 2 \times 3 - 1 \times (-5) + 3 \times (-7)$$

$$= 6 + 5 - 21 = 11 - 21 = -10$$ **(1)**

7. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\vec{a} - \vec{b})$ is a unit vector.

Delhi 2014C

Given, $\vec{a}$ and $\vec{b}$ are two unit vectors, then $|\vec{a}| = |\vec{b}| = 1$

Also, $(\sqrt{3}\vec{a} - \vec{b})$ is a unit vector.

$\therefore \quad |\sqrt{3}\vec{a} - \vec{b}| = 1 \Rightarrow |\sqrt{3}\vec{a} - \vec{b}|^2 = 1^2$

$$\Rightarrow (\sqrt{3}\vec{a} - \vec{b}) \cdot (\sqrt{3}\vec{a} - \vec{b}) = 1 \quad [\because |\vec{a}|^2 = \vec{a} \cdot \vec{a}]$$

$$\Rightarrow 3(\vec{a} \cdot \vec{a}) - \sqrt{3}(\vec{a} \cdot \vec{b}) - \sqrt{3}(\vec{b} \cdot \vec{a}) + \vec{b} \cdot \vec{b} = 1$$

$$\Rightarrow 3|\vec{a}|^2 - \sqrt{3}|\vec{a}||\vec{b}|\cos\theta$$

$$- \sqrt{3}|\vec{b}||\vec{a}|\cos\theta + |\vec{b}|^2 = 1 \qquad \textbf{(1/2)}$$

[let θ be the angle between $\vec{a}$ and $\vec{b}$]

$$\Rightarrow \quad 3 - \sqrt{3}\cos\theta - \sqrt{3}\cos\theta + 1 = 1$$

$$\Rightarrow \quad 3 = 2\sqrt{3}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{3}{2\sqrt{3}} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Hence, required angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$.

8. If $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$, find the angle between $\vec{a}$ and $\vec{b}$. **All India 2014C**

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Given, $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$

We know that, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

$$\therefore \quad |\vec{a}||\vec{b}|\sin\theta = 12$$

$$\Rightarrow \quad \sin\theta = \frac{12}{|\vec{a}||\vec{b}|} \Rightarrow \sin\theta = \frac{12}{8 \times 3}$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

Hence, the required angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$. **(1)**

9. Write the projection of the vector $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ on the vector $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$.

Delhi 2014C

Given, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

Then, projection of $\vec{a}$ on $\vec{b}$ is given by

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \left[\frac{(2\hat{i} - \hat{j} + \hat{k})(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{(1)^2 + (2)^2 + (2)^2}}\right]$$

$$= \frac{2 \times 1 + (-1) \times (2) + 1 \times 2}{\sqrt{1 + 4 + 4}}$$

$$= \frac{2 - 2 + 2}{\sqrt{9}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

Hence, the projection of vector $(2\hat{i} - \hat{j} + \hat{k})$ on $(\hat{i} + 2\hat{j} + 2\hat{k})$ is $\frac{2}{3}$. **(1)**

10. Write the value of λ, so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other. **Delhi 2013C, 2008**

Given, vectors are $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$

and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

Since, vectors are perpendicular.

$\therefore \quad \vec{a} \cdot \vec{b} = 0$ **(1/2)**

$\Rightarrow \quad (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 2 - 2\lambda + 3 = 0$

$\therefore \quad \lambda = 5/2$ **(1/2)**

11. Write the projection of the vector $7\hat{i} + \hat{j} - 4\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$. **Delhi 2013C**

Let $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

$\therefore$ Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

$$= \frac{(7\hat{i} + \hat{j} - 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})}{|2\hat{i} + 6\hat{j} + 3\hat{k}|}$$

$$= \frac{14 + 6 - 12}{\sqrt{(2)^2 + (6)^2 + (3)^2}}$$

$$= \frac{8}{\sqrt{4 + 36 + 9}} = \frac{8}{\sqrt{49}} = \frac{8}{7}$$ **(1)**

12. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{a}|$, then prove that vector $2\vec{a} + \vec{b}$ is perpendicular to vector $\vec{b}$. **HOTS; Delhi 2013**

To prove, $(2\vec{a} + \vec{b}) \perp \vec{b}$

Given, $|\vec{a} + \vec{b}| = |\vec{a}|$

On squaring both sides, we get

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2$$

$$\Rightarrow \quad |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{a}|^2$$

$$\Rightarrow \quad 2\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} = 0 \quad [\because |\vec{x}|^2 = \vec{x} \cdot \vec{x} = \vec{x}^2]$$

$$\Rightarrow \quad (2\vec{a} + \vec{b}) \cdot \vec{b} = 0 \Rightarrow (2\vec{a} + \vec{b}) \perp \vec{b}$$

$[\because \text{If } \vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \cdot \vec{b} = 0]$ **(1)**

Hence proved.

13. Find $|\vec{x}|$, if for a unit vector $\hat{a}$, $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 15$. **HOTS; All India 2013**

Given, $\hat{a}$ is a unit vector. Then, $|\hat{a}| = 1$

Now, we have $(\vec{x} - \hat{a}) \cdot (\vec{x} + \hat{a}) = 15$

$\Rightarrow \quad \vec{x} \cdot \vec{x} - \hat{a} \cdot \vec{x} + \vec{x} \cdot \hat{a} - \hat{a} \cdot \hat{a} = 15$

$\Rightarrow \quad \vec{x} \cdot \vec{x} - \hat{a} \cdot \vec{x} + \hat{a} \cdot \vec{x} - \hat{a}.\hat{a} = 15$

[$\because$ scalar product is commutative]

$\Rightarrow \quad |\vec{x}|^2 - |\hat{a}|^2 = 15 \quad [\because \vec{z} \cdot \vec{z} = |\vec{z}|^2]$

$\Rightarrow \quad |\vec{x}|^2 - 1 = 15 \Rightarrow |\vec{x}|^2 = 16$

$\therefore \quad |\vec{x}| = 4$ **(1)**

14. Find λ, when projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

HOTS; Delhi 2012

Given, $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ and projection of $\vec{a}$ on $\vec{b} = 4$

$$\Rightarrow \quad \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 4$$

$$\left[\because \text{projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}\right]$$

$$\Rightarrow \quad \frac{(\lambda\hat{i} + \hat{j} + 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = 4$$

$$\Rightarrow \quad \frac{2\lambda + 6 + 12}{\sqrt{49}} = 4 \Rightarrow 2\lambda + 18 = 28$$

$\therefore \quad \lambda = 5$ **(1)**

15. Write the value of $(\hat{k} \times \hat{j}) \cdot \hat{i} + \hat{j} \cdot \hat{k}$.

HOTS; All India 2012

Use the results $\hat{j} \times \hat{k} = \hat{i}$,
$\hat{j} \cdot \hat{k} = 0$ and $\hat{i} \cdot \hat{i} = 1$

We have, $(\hat{k} \times \hat{j}) \cdot \hat{i} + \hat{j} \cdot \hat{k} = (-\hat{i}) \cdot \hat{i} + \hat{j} \cdot \hat{k}$

$[\because \hat{j} \times \hat{k} = \hat{i} \Rightarrow \hat{k} \times \hat{j} = -\hat{i}$ and $\hat{j} \cdot \hat{k} = 0]$

$= -\hat{i}^2 + 0 = -1 \quad [\because \hat{i}^2 = 1]$ **(1)**

16. If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector $\vec{b}$? **Foreign 2011**

Given, $\vec{a} \cdot \vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$ ...(i)

and $\vec{a} \cdot \vec{b} = 0$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = 0$ ...(ii)

From Eqs. (i) and (ii), it may be concluded that $\vec{b}$ is either zero or non-zero perpendicular vector.

(1)

17. Write the projection of vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$. **All India 2011**

Let given vector are $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

Projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{(\hat{i} - \hat{j}).(\hat{i} + \hat{j})}{\sqrt{(1)^2 + (1)^2}}$$

$$= \frac{1-1}{\sqrt{2}} = \frac{0}{\sqrt{2}} = 0 \quad \begin{bmatrix} \because \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1 \\ \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0 \end{bmatrix}$$ **(1)**

18. Write the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, having $\vec{a} \cdot \vec{b} = \sqrt{6}$. **All India 2011**

Let θ be the angle between $\vec{a}$ and $\vec{b}$, then use the following formula

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Given, $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = \sqrt{6}$

Now, angle between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{\sqrt{3}\times\sqrt{2}}{\sqrt{3}\times 2} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\therefore \quad \theta = \frac{\pi}{4} \qquad \left[\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right] \text{ (1)}$$

19. For what value of λ are the vectors $\hat{i} + 2\lambda\hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} - 3\hat{k}$ perpendicular?

All India 2011C

If two vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a}\cdot\vec{b} = 0$

Given vectors ore $(\hat{i} + 2\lambda\hat{j} + \hat{k})$

and $(2\hat{i} + \hat{j} - 3\hat{k})$.

Also given, the vectors are perpendicular, so their dot product is zero.

$$\therefore \quad (\hat{i} + 2\lambda\hat{j} + \hat{k})\cdot(2\hat{i} + \hat{j} - 3\hat{k}) = 0$$

$$\Rightarrow \quad 2 + 2\lambda - 3 = 0$$

$$\Rightarrow \quad 2\lambda - 1 = 0$$

$$\Rightarrow \quad 2\lambda = 1 \text{ or } \lambda = \frac{1}{2} \qquad \text{(1)}$$

Hence, required value of λ is 1/2.

> If two vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then $\vec{a} \cdot \vec{b} = 0$

Given vectors ore $(\hat{i} + 2\lambda\hat{j} + \hat{k})$

and $(2\hat{i} + \hat{j} - 3\hat{k})$.

Also given, the vectors are perpendicular, so their dot product is zero.

$\therefore \quad (\hat{i} + 2\lambda\hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = 0$

$\Rightarrow \quad 2 + 2\lambda - 3 = 0$

$\Rightarrow \quad 2\lambda - 1 = 0$

$\Rightarrow \quad 2\lambda = 1$ or $\lambda = \frac{1}{2}$ **(1)**

Hence, required value of λ is 1/2.

20. If $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°, then find $\vec{a} \cdot \vec{b}$. **Delhi 2011C**

We know that, $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos\theta$

On putting $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and $\theta = 60°$,

we get

$\vec{a} . \vec{b} = \sqrt{3} \times 2 \cos 60°$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = \frac{1}{2} \times 2\sqrt{3} = \sqrt{3} \quad \left[\because \cos 60° = \frac{1}{2}\right]$

$\therefore \quad \vec{a} \cdot \vec{b} = \sqrt{3}$ **(1)**

21. Find the value of λ, if the vectors $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other. **All India 2010C**

Do same as Que. 19. **[Ans.** $\lambda = 3$**]**

22. If $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 3$, then find the projection of $\vec{b}$ on $\vec{a}$. **All India 2010C**

Given, $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 3$

$\therefore$ Projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|}$

$= \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ $\quad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

$= \dfrac{3}{2}$ $[\because \vec{a} \cdot \vec{b} = 3$ and $|\vec{a}| = 2]$ **(1)**

23. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2/3$ and $\vec{a} \times \vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$. **All India 2010**

Do same as Que. 5. **[Ans.** $\dfrac{\pi}{3}$**]**

24. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, then find the angle between $\vec{a} \times \vec{b}$. **HOTS; All India 2010**

Use the following formulae:

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

and $$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\cdot\hat{n}$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$.

Given, $|\vec{a}\cdot\vec{b}| = |\vec{a}\times\vec{b}|$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = |\vec{a}||\vec{b}|\sin\theta \quad [\because |\hat{n}| = 1]$

$\Rightarrow \quad \cos\theta = \sin\theta$

On dividing both sides by $\cos\theta$, we get

$\tan\theta = 1$

$\Rightarrow \quad \tan\theta = \tan\frac{\pi}{4} \quad \left[\because 1 = \tan\frac{\pi}{4}\right]$

$\therefore \quad \theta = \frac{\pi}{4}$

So, angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$. **(1)**

25. Find λ, if $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$.

All India 2010

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Given, $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k})$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 14 \\ 1 & -\lambda & 7 \end{vmatrix} = \vec{0}$$

$\Rightarrow \hat{i}(42 + 14\lambda) - \hat{j}(14 - 14) + \hat{k}(-2\lambda - 6) = \vec{0}$

$\Rightarrow \hat{i}(42 + 14\lambda) + \hat{k}(-2\lambda - 6) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

$[\because \vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}]$

On comparing coefficients of $\hat{i}$ and $\hat{k}$ from both sides, we get

$42 + 14\lambda = 0$

$\Rightarrow \lambda = -3$

and $-2\lambda - 6 = 0$

$\Rightarrow \lambda = -3$ **(1)**

Hence, required value of λ is -3.

26. Find $\vec{a} \cdot \vec{b}$, if $\vec{a} = -\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$. **All India 2009C**

Given, $\vec{a} = -\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{b} = 2\hat{i} + 3\hat{j} - \hat{k}$

Then, $\vec{a} \cdot \vec{b} = (-\hat{i} + \hat{j} - 2\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k})$

$= -2 + 3 + 2 = 3$ **(1)**

27. Find $\vec{a} \cdot \vec{b}$, if $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + 3\hat{k}$. **Delhi 2009C**

Do same as Que. 26. [**Ans.** 9]

28. Find the value of P, if

$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + 3\hat{j} + P\hat{k}) = \vec{0}$. **All India 2009**

Do same as Que. 25. $\left[\textbf{Ans. } \frac{27}{2}\right]$

29. If $\hat{P}$ is a unit vector and $(\vec{x} - \hat{P}) \cdot (\vec{x} + \hat{P}) = 80$, then find $|\vec{x}|$. **HOTS; All India 2009**

Do same as Que. 13. **[Ans.** 9]

30. Find the angle between $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively, when $|\vec{a} \times \vec{b}| = \sqrt{3}$. **Delhi 2009**

Given, $|\vec{a}| = 1, |\vec{b}| = 2$ and $|\vec{a} \times \vec{b}| = \sqrt{3}$

$\Rightarrow \quad |\vec{a}||\vec{b}| \sin\theta = \sqrt{3}$

$[\because \vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \cdot \hat{n} \text{ and } |\hat{n}| = 1]$

$\Rightarrow \quad 1 \times 2 \times \sin\theta = \sqrt{3} \quad [\because |\vec{a}| = 1 \text{ and } |\vec{b}| = 2]$

$\Rightarrow \quad \sin\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

Hence, angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$. **(1)**

31. Write the value of P, for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + P\hat{j} + 3\hat{k}$ are parallel vectors. **Delhi 2009**

Given vectors are $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + P\hat{j} + 3\hat{k}$.

Also, $\vec{a}$ and $\vec{b}$ are parallel vectors.

So, $\vec{a} \times \vec{b} = 0$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & P & 3 \end{vmatrix} = \vec{0}$$

$$\Rightarrow \hat{i}\,(6 - 9P) - \hat{j}\,(9 - 9) + \hat{k}\,(3P - 2) = \vec{0}$$

$$\Rightarrow \hat{i}\,(6 - 9P) + \hat{k}\,(3P - 2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

On comparing the coefficients of $\hat{i}$ or $\hat{k}$ from both sides, we get

$$6 - 9P = 0 \Rightarrow P = \frac{2}{3}$$ **(1)**

Alternate Method

> If the two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel to each other, then use the following relation.
>
> $$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Given, $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + P\hat{j} + 3\hat{k}$ are parallel vectors.

Then, $\frac{3}{1} = \frac{2}{P} = \frac{9}{3} \Rightarrow P = \frac{2}{3}$ **(1)**

32. Find the projection of $\vec{a}$ on $\vec{b}$, if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$. **Delhi 2009**

Do same as Que. 9. $\left[\textbf{Ans. } \frac{8}{7}\right]$

33. Find value of the following:

$$\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j}).$$

HOTS; All India 2008C

We have, $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{i}\times\hat{k})+\hat{k}\cdot(\hat{i}\times\hat{j})$

$$=\hat{i}\cdot\hat{i}+\hat{j}\cdot(-\hat{j})+\hat{k}\cdot\hat{k}$$

$$\left[\begin{array}{ll}\because\ \hat{i}\times\hat{j}=\hat{k}; & \hat{j}\times\hat{k}=\hat{i}\\ \hat{k}\times\hat{i}=\hat{j}\ \Rightarrow & \hat{i}\times\hat{k}=-\hat{j}\end{array}\right]$$

$$=\hat{i}^2-\hat{j}^2+\hat{k}^2$$

$$=1-1+1=1$$ **(1)**

34. Find $|\vec{a}\times\vec{b}|$, if $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$. **Delhi 2008C**

Given, $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$

$$\therefore\ \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -7 & 7\\ 3 & -2 & 2\end{vmatrix}$$

$$=\hat{i}(-14+14)-\hat{j}(2-21)+\hat{k}(-2+21)$$

$$=19\hat{j}+19\hat{k}$$

Now, $|\vec{a}\times\vec{b}|=\sqrt{(19)^2+(19)^2}$

$$=\sqrt{2(19)^2}=19\sqrt{2}$$ **(1)**

35. If $|\vec{a}|=\sqrt{3}, |\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=3$, then find the angle between $\vec{a}$ and $\vec{b}$. **All India 2008**

Do same as Que. 18. $\left[\textbf{Ans. } \frac{\pi}{6}\right]$

36. Find angle between vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$. **Delhi 2008**

Given, vectors are

$$\vec{a} = \hat{i} - \hat{j} + \hat{k} \text{ and } \hat{b} = \hat{i} + \hat{j} - \hat{k}.$$

Then, $\vec{a} \cdot \vec{b} = (\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k})$

$$= 1 - 1 - 1 = -1$$

$$|\vec{a}| = \sqrt{(1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$$

and $|\vec{b}| = \sqrt{(1)^2 + (1)^2 + (-1)^2} = \sqrt{3}$

We know that, angle between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\therefore \quad \cos\theta = \frac{-1}{\sqrt{3} \times \sqrt{3}} = -\frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

which is the required angle between $\vec{a}$ and $\vec{b}$. **(1)**

4 Marks Questions

37. Prove that, for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,

$$[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a} \quad \vec{b} \quad \vec{c}]$$ **Delhi 2014**

If use the property that in a scalar triple product, if any two vectors are equal, then value of scalar triple product will be zero and $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]$.

We have, LHS $= [\vec{a} + \vec{b}\ \ \vec{b} + \vec{c}\ \ \vec{c} + \vec{a}]$

$= (\vec{a} + \vec{b}) \cdot [(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})]$

$= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a})$

$= (\vec{a} + \vec{b}) \cdot (\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a})$

$[\because \vec{c} \times \vec{c} = 0]$ **(2)**

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a})$

$+ \vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{a}]$

$+ [\vec{b}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{b}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{a}]$

$= 2\ [\vec{a}\ \vec{b}\ \vec{c}] =$ RHS **Hence proved. (2)**

38. Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

All India 2008; Delhi 2014, 2008

Firstly, write the given expression $\vec{a} + \vec{b} + \vec{c} = 0$ as $\vec{a} + \vec{b} = -\vec{c}$ and then square both sides and symplify to get the angle between $\vec{a}$ and $\vec{b}$.

Given, $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$

Also, $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow (\vec{a} + \vec{b})^2 = (-\vec{c})^2$ [squaring on both sides]

$\Rightarrow (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = (-\vec{c}) \cdot (-\vec{c})$

$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = \vec{c} \cdot \vec{c}$

$\Rightarrow |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{c}|^2$

[$\because \vec{x} \cdot \vec{x} = |\vec{x}|^2$ and $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$] **(1)**

$\Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = |\vec{c}|^2$

$$\Rightarrow \quad |\vec{a}|^2 + 2|\vec{a}|\cdot|\vec{b}|\cos\theta$$

$$+|\vec{b}|^2 = |\vec{c}|^2 \quad \text{...(i)} \ \mathbf{(1)}$$

$$[\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta]$$

$$\Rightarrow (3)^2 + 2\times 3\times 5\cos\theta + (5)^2 = (7)^2$$

$$[\because |\vec{a}| = 3, |\vec{b}| = 5 \text{ and } |\vec{c}| = 7]$$

$$\Rightarrow \quad 9 + 30\cos\theta + 25 = 49 \quad \mathbf{(1)}$$

$$\Rightarrow \quad 30\cos\theta = 49 - 34$$

$$\Rightarrow \quad \cos\theta = \frac{15}{30} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \left[\because \frac{1}{2} = \cos\frac{\pi}{3}\right]$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$. **(1)**

39. Show that the four points A, B, C and D with position vectors $4\hat{i} + 5\hat{j} + \hat{k}, -j - k$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $4(-\hat{i} + \hat{j} + \hat{k})$ respectively are coplanar. **All India 2014**

Given, points are $A = 4\hat{i} + 5\hat{j} + \hat{k}, B = -\hat{j} - \hat{k}$,
$C = 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $D = 4(-\hat{i} + \hat{j} + \hat{k})$.

We know that, the four points A, B, C, and D will be coplanar, if the three vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AB}$ are coplanar, i.e. if

$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$$

$$\therefore \quad \overrightarrow{AB} = -\hat{j} - \hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -4\hat{i} - 6\hat{j} - 2\hat{k} \quad \textbf{(1)}$$

$$\overrightarrow{AC} = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -\hat{i} + 4\hat{j} + 3\hat{k}$$

and $$\overrightarrow{AD} = 4(-\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -8\hat{i} - \hat{j} + 3\hat{k} \quad \textbf{(1)}$$

Now, $$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} \quad \textbf{(1)}$$

$$= -4(12 + 3) + 6(-3 + 24) - 2(1 + 32)$$
$$= -60 + 126 - 66 = -126 + 126 = 0$$

Hence, points A, B, C and D are coplanar. **(1)**

40. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ and hence, find the unit vector along $\vec{b} + \vec{c}$. **All India 2014**

Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and
$\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$.

Now, $\vec{b} + \vec{c} = 2\hat{i} + 4\hat{j} - 5\hat{k} + \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$$= (2 + \lambda)\, i + 6j - 2k$$

$$\therefore \quad |\vec{b} + \vec{c}| = \sqrt{(2+\lambda)^2 + (6)^2 + (-2)^2}$$

$$= \sqrt{4 + \lambda^2 + 4\lambda + 36 + 4}$$

$$= \sqrt{\lambda^2 + 4\lambda + 44} \qquad \textbf{(1)}$$

The unit vector along $\vec{b} + \vec{c}$

$$= \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}$$

$$= \frac{(2+\lambda)\,\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} \qquad \text{...(i)}$$

Given, scalar product of $(\hat{i} + \hat{j} + \hat{k})$ with this unit vector is 1. **(1)**

$$\therefore \quad (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|} = 1$$

$$\Rightarrow \quad (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{(2+\lambda)\,\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \quad \frac{1(2+\lambda) + 1(6) + 1(-2)}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \quad \frac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$$

$$\Rightarrow \quad \lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44}$$

$$\Rightarrow \quad (\lambda + 6)^2 = \lambda^2 + 4\lambda + 44$$

[squaring on both sides]

$$\Rightarrow \quad \lambda^2 + 36 + 12\lambda = \lambda^2 + 4\lambda + 44$$

$$\Rightarrow \quad 8\lambda = 8$$

$$\Rightarrow \qquad \lambda = 1$$

Hence, the value of λ is 1.

On substituting the value of λ in Eq. (i), we get

Unit vector along $\vec{b} + \vec{c}$

$$= \frac{(2+1)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(1)^2 + 4(1) + 44}} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{1 + 4 + 44}}$$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{49}} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k} \qquad \textbf{(1)}$$

41. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha} = 4\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{\beta} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{p} \cdot \vec{q} = 21$, where $\vec{q} = 3\hat{i} + \hat{j} - \hat{k}$. **All India 2014C**

Given $\alpha = 4\hat{i} + 5\hat{j} - \hat{k}, \beta = \hat{i} - 4\hat{j} + 5\hat{k}$

$$q = 3\hat{i} + \hat{j} - \hat{k}$$

Also, vector $\vec{p}$ is perpendicular to α and β.

Then, $\vec{p} = \lambda\,(\vec{\alpha} \times \vec{\beta})$

$$\text{Now, } \vec{\alpha} \times \vec{\beta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= \hat{i}\,(25 - 4) - \hat{j}\,(20 + 1) + \hat{k}\,(-16 - 5)$$

$$= \hat{i}\,(21) - \hat{j}\,(21) + \hat{k}\,(-21)$$

$$= 21\hat{i} - 21\hat{j} - 21\hat{k}$$

So, $\vec{p} = 21\lambda\hat{i} - 21\lambda\hat{j} - 21\lambda\hat{k}$ ...(i)

Also, given that $\vec{p} \cdot \vec{q} = 21$

$\therefore \quad (21\lambda\hat{i} - 21\lambda\hat{j} - 21\lambda\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 21$

$\Rightarrow \quad 63\lambda - 21\lambda + 21\lambda = 21$

$\Rightarrow \quad 63\lambda = 21 \Rightarrow \lambda = 1/3$ **(1)**

On putting $\lambda = \frac{1}{3}$ in Eq. (i), we get

$$\vec{p} = 21 \times \frac{1}{3}\hat{i} - 21 \times \frac{1}{3}\hat{j} - 21 \times \frac{1}{3}\hat{k}$$

$\Rightarrow \quad \vec{p} = 7\hat{i} - 7\hat{j} - 7\hat{k}$

which is the required vector. **(1)**

42. Find a unit vector perpendicular to both of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Foreign 2014

Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

Let the required unit vector be

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$$

then $$\sqrt{x^2 + y^2 + z^2} = 1$$

$\Rightarrow \quad x^2 + y^2 + z^2 = 1 \quad$ ...(i)

Now, $\quad \vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\vec{a} - \vec{b} = (\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) - \hat{j} - 2\hat{k}$

(1)

Since, $\vec{r}$ is perpendicular to $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$,

$\therefore \quad \vec{r} \cdot (\vec{a} + \vec{b}) = 0 \quad$ and $\quad \vec{r} \cdot (\vec{a} - \vec{b}) = 0$

i.e. $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0$

$\Rightarrow \quad 2x + 3y + 4z = 0 \quad$ ...(ii)

and $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-\hat{j} - 2\hat{k}) = 0$

$\Rightarrow \quad -y - 2z = 0$

$\Rightarrow \quad y = -2z \quad$ ...(iii)

On putting the value of y in Eq. (ii), we get

$2x + 3(-2z) + 4z = 0$

$\Rightarrow \quad x = z \quad$ **(1)**

On substituting the value of x and y in Eq. (i), we get

$z^2 + 4z^2 + z^2 = 1 \Rightarrow z = \pm \frac{1}{\sqrt{6}}$ and

then, $\quad x = \pm \frac{1}{\sqrt{6}}$ and $y = \mp \frac{2}{\sqrt{6}}$ **(1)**

Hence, the required vectors are

$\frac{1}{\sqrt{6}}\hat{i} - \frac{2}{\sqrt{6}}\hat{j} + \frac{1}{\sqrt{6}}\hat{k}$

and $\quad \frac{-1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}.$ **(1)**

43. Find the unit vector perpendicular to the plane ABC where the position vectors of A, B and C are $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} + 3\hat{k}$ respectively. **All India 2014C**

A unit vector perpendicular to plane ABC is

$$\frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}$$

Let O be the origin of reference.

Then, given $\overrightarrow{OA} = 2\hat{i} - \hat{j} + \hat{k}$,

$\overrightarrow{OB} = \hat{i} + \hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = 2\hat{i} + 3\hat{k}$

$\therefore \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} + \hat{j} + 2\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= -\hat{i} + 2\hat{j} + \hat{k}$ **(1)**

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\hat{i} + 3\hat{k} - 2\hat{i} + \hat{j} - \hat{k}$

$= \hat{j} + 2\hat{k}$

$$\text{Now, } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$= \hat{i}(4-1) - \hat{j}(-2-0) + \hat{k}(-1-0)$

$= 3\hat{i} + 2\hat{j} - \hat{k}$ **(1)**

Then, $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(3)^2 + (2)^2 + (-1)^2}$

$= \sqrt{9+4+1} = \sqrt{14}$ **(1)**

Unit vector perpendicular to the plane ABC

$$= \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} = \frac{3\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{14}}$$

$$= \frac{3}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} - \frac{1}{\sqrt{14}}\hat{k}$$ **(1)**

44. Show that the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, if and only if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar. **Foreign 2014**

Consider,

$$[(\vec{a}+\vec{b})\,(\vec{b}+\vec{c})\,(\vec{c}+\vec{a})]$$

$$=(\vec{a}+\vec{b})\cdot\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\}$$

$$=(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$$

$$=(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a})$$

$$\because \vec{c}\times\vec{c}=\vec{0} \quad \textbf{(2)}$$

$$=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{b}\times\vec{a})+\vec{a}\cdot(\vec{c}\times\vec{a})$$
$$+\vec{b}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{b}\times\vec{a})+\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$=\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{b}\cdot(\vec{c}\times\vec{a})$$

$$[\because [\vec{a}\ \vec{b}\ \vec{a}]=[\vec{b}\ \vec{b}\ \vec{a}]=[\vec{a}\ \vec{c}\ \vec{a}]=0]$$

$$=2\,[\vec{a}\ \vec{b}\ \vec{c}]$$

Now, we can see that

$$[(\vec{a}+\vec{b})\,(\vec{b}+\vec{c})\,(\vec{c}+\vec{a})]=2\,[\vec{a}\ \vec{b}\ \vec{c}]$$

Hence, the vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if and only if $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are coplanar **(2)**

45. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, then prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

HOTS; Delhi 2013C, 2011

If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular to each other, then $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$ and if all three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are equally inclined with the vector $(\vec{a} + \vec{b} + \vec{c})$, that means each vector $\vec{a}, \vec{b}$ and $\vec{c}$ makes equal angle with $(\vec{a} + \vec{b} + \vec{c})$ by using formula $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

Given, $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ [say]

and $\vec{a} \cdot \vec{b} = 0, \vec{b} \cdot \vec{c} = 0$ and $\vec{c} \cdot \vec{a} = 0$ **(1/2)**

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$

$= \lambda^2 + \lambda^2 + \lambda^2 + 2(0 + 0 + 0) = 3\lambda^2$

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = \pm\sqrt{3}\lambda$ **(1)**

Suppose $(\vec{a} + \vec{b} + \vec{c})$ is inclined at angles θ_1, θ_2 and θ_3 respectively with vectors $\vec{a}, \vec{b}$ and $\vec{c}$, then

$(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a} = |\vec{a} + \vec{b} + \vec{c}| \,.\, |\vec{a}| \cos\theta_1$

$[\because \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta]$

$\Rightarrow \quad |\vec{a}|^2 + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = \pm\sqrt{3}\lambda \times \lambda \cos\theta_1$

$\Rightarrow \quad \lambda^2 + 0 + 0 = \pm\sqrt{3}\lambda^2 \cos\theta_1$

$\therefore \quad \cos\theta_1 = \pm\frac{1}{\sqrt{3}}$

$(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{b} = |\vec{a} + \vec{b} + \vec{c}||\vec{b}| \cdot \cos\theta_2$ **(1)**

$$\Rightarrow \quad \vec{a}\cdot\vec{b} + |\vec{b}|^2 + \vec{b}.\vec{c} = \pm\sqrt{3}\,\lambda\cdot\lambda\cos\theta_2$$

$$\Rightarrow \quad 0 + \lambda^2 + 0 = \pm\sqrt{3}\,\lambda^2\cos\theta_2$$

$$\Rightarrow \quad \cos\theta_2 = \pm\frac{1}{\sqrt{3}}$$

Similarly, $(\vec{a}+\vec{b}+\vec{c}).\vec{c} = |\vec{a}+\vec{b}+\vec{c}||\vec{c}|\cos\theta_3$

$$\Rightarrow \quad \cos\theta_1 = \pm\frac{1}{\sqrt{3}} \qquad \textbf{(1)}$$

Thus, $\cos\theta_1 = \cos\theta_2 = \cos\theta_3 = \pm\frac{1}{\sqrt{3}}$

Hence, it is proved that $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. **(1/2)**

46. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$, then find a vector $\vec{c}$, such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}\cdot\vec{c} = 3$.

Delhi 2013, 2008

> If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors, then
>
> $$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$
>
> and $$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$

and $\quad \vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore \quad \vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$$

$$= \hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x) \quad \textbf{(1)}$$

Now, $\vec{a} \times \vec{c} = \vec{b}$ [given]

$$\Rightarrow \quad \hat{i}(z-y) + \hat{j}(x-z) + \hat{k}(y-x)$$

$$= 0\hat{i} + 1\hat{j} + (-1)\hat{k} \qquad [\because \vec{b} = \hat{j} - \hat{k}]$$

On comparing the coefficients from both sides, we get

$$z - y = 0,\ x - z = 1,\ y - x = -1$$

$$\Rightarrow \quad y = z \text{ and } x - y = 1 \quad \text{...(i)}$$

Also given, $\vec{a} \cdot \vec{c} = 3$

$$\Rightarrow \quad (\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$$

$$\Rightarrow \quad x + y + z = 3 \quad \textbf{(1)}$$

$$\Rightarrow \quad x + 2y = 3 [\because y = z] \text{...(ii)}$$

On solving Eqs. (i) and (ii), we get

$$3y = 2$$

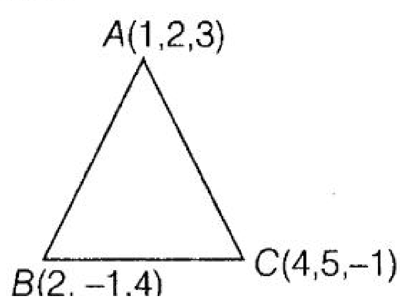
$$\Rightarrow \quad y = \frac{2}{3} = z \qquad [\because y = z]$$

From Eq. (i) $\quad x = 1 + y = 1 + \frac{2}{3} = \frac{5}{3}$ **(1)**

Hence, $\quad \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$ **(1)**

47. Using vectors, find the area of the ΔABC, whose vertices are $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$. **All India 2013**

Let the position vectors of the vertices A, B and C of ΔABC be **(1)**



$$\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k},\ \overrightarrow{OB} = 2\hat{i} - \hat{j} + 4\hat{k}$$

and $\overrightarrow{OC} = 4\hat{i} + 5\hat{j} - \hat{k}$, respectively.

Then, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= \hat{i} - 3\hat{j} + \hat{k}$$

and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= (3\hat{i} + 3\hat{j} - 4\hat{k})$$ **(1)**

Then, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$

$$= \hat{i}\,(12 - 3) - \hat{j}\,(-4 - 3) + \hat{k}\,(3 + 9)$$

$$= 9\hat{i} + 7\hat{j} + 12\,\hat{k}$$ **(1)**

$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(9)^2 + (7)^2 + (12)^2}$

$$= \sqrt{81 + 49 + 144} = \sqrt{274}$$

Hence, area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\sqrt{274} \text{ sq units}$$ **(1)**

48. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ, so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors. **All India 2013**

Use the result that if $\vec{a}$ and $\vec{b}$ are perpendicular, then their dot product should be zero and simplify it.

Given, $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$

Then, $\vec{a} + \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) + (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}$ **(1)**

and $\vec{a} - \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) - (5\hat{i} - \hat{j} + \lambda\hat{k})$

$= -4\hat{i} + (7-\lambda)\hat{k}$ **(1)**

Since, $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular vectors, then $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$\Rightarrow [6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}] \cdot [-4\hat{i} + (7-\lambda)\hat{k}] = 0$ **(1)**

$\Rightarrow \quad -24 + (7+\lambda)(7-\lambda) = 0$

$\Rightarrow \quad 49 - \lambda^2 = 24 \Rightarrow \lambda^2 = 25$

$\therefore \quad \lambda = \pm 5$ **(1)**

49. If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ, so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors. **All India 2013**

Do same as Que. 48. [**Ans.** $\lambda = \pm 1$]

50. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors, such that $|\vec{a}| = 5, |\vec{b}| = 12, |\vec{c}| = 13$ and $\vec{a} + \vec{b} + \vec{c} = 0$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. **Delhi 2012**

Use the following formula:

$$(\vec{a}+\vec{b}+\vec{c})^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a})$$

Given, $|\vec{a}| = 5, |\vec{b}| = 12, |\vec{c}| = 13$

and $\vec{a}+\vec{b}+\vec{c} = 0$

On multiplying both sides by $(\vec{a}+\vec{b}+\vec{c})$, we get

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c}) = \vec{0}\cdot(\vec{a}+\vec{b}+\vec{c}) \quad \textbf{(1)}$$

$$\Rightarrow \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + \vec{c}\cdot\vec{c} = 0$$

$$\Rightarrow |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + |\vec{b}|^2 + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + |\vec{c}|^2 = 0$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

(1½)

$[\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}, \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{b}, \vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{c}$ and $\vec{x}\cdot\vec{x} = |\vec{x}|^2]$

$$\Rightarrow (5)^2 + (12)^2 + (13)^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$$

$[\because |\vec{a}| = 5, |\vec{b}| = 12$ and $|\vec{c}| = 13]$

$$\Rightarrow 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -338 \quad \textbf{(1½)}$$

Hence, $\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -169$

51. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{p}$, which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p} \cdot \vec{c} = 18$.

HOTS; All India 2012, 2010

Given, vectors are $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$,

$$\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$$

and $$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$$

We have to find a vector $\vec{p}$, such that

$$\vec{p} \cdot \vec{a} = 0 \quad \text{...(i)}$$

and $$\vec{p} \cdot \vec{b} = 0 \quad \text{...(ii)}$$

[$\because \vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, given]

and $$\vec{p} \cdot \vec{c} = 18 \quad \text{...(iii)}$$ **(1)**

So, let $$\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$$

From Eqs. (i), (ii) and (iii), we get

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + 4\hat{j} + 2\hat{k}) = 0$$

$$\Rightarrow \quad x + 4y + 2z = 0 \quad \text{...(iv)}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 7\hat{k}) = 0$$

$$\Rightarrow \quad 3x - 2y + 7z = 0 \quad \text{...(v)}$$

and $$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - \hat{j} + 4\hat{k}) = 0$$

$$\Rightarrow \quad 2x - y + 4z = 18 \quad \text{...(vi)}$$

On multiplying Eq. (iv) by 3 and subtracting it from Eq. (v), we get

$$-14y + z = 0 \quad \text{...(vii)}$$

Now, multiplying Eq. (iv) by 2 and subtracting it from Eq. (vi), we get

$$-9y = 18 \quad \Rightarrow \quad y = -2$$ **(1)**

On putting $y = -2$ in Eq. (vii), we get

$$-14(-2) + z = 0$$

$$\Rightarrow \quad 28 + z = 0$$

$$\Rightarrow \quad z = -28$$

On putting $y = -2$ and $z = -28$ in Eq. (iv), we get

$$x + 4(-2) + 2(-28) = 0$$

$$\Rightarrow \quad x - 8 - 56 = 0$$

$$\Rightarrow \quad x = 64$$ **(1½)**

Hence, the required vector

$$\vec{p} = x\hat{i} + y\hat{j} + z\hat{k} \text{ is}$$

$$\vec{p} = 64\hat{i} - 2\hat{j} - 28\hat{k}$$ **(1/2)**

52. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$. **Delhi 2011**

> Use the concept, if $\vec{a}$ and $\vec{b}$ are two vectors, then a unit vector perpendicular to both of them $= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Given, $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$

and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Then, $\vec{a} + \vec{b} = (3\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - 2\hat{k})$

$$= 4\hat{i} + 4\hat{j}$$

and $\vec{a} - \vec{b} = (3\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - 2\hat{k})$

$$= 2\hat{i} + 4\hat{k}$$ **(1)**

Let $\vec{a} + \vec{b} = \vec{c}$ and $\vec{a} - \vec{b} = \vec{d}$, so that we have

$c = 4i + 4j$ and $d = 2i + 4k$.

$$\text{Now, } \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= \hat{i}(16-0) - \hat{j}(16-0) + \hat{k}(0-8)$$

$$\Rightarrow \quad \vec{c} \times \vec{d} = 16\hat{i} - 16\hat{j} - 8\hat{k} \quad \text{...(i)} \ \mathbf{(1)}$$

$$\therefore \quad |\vec{c} \times \vec{d}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{256 + 256 + 64}$$

$$= \sqrt{576} = 24 \quad \text{...(ii)} \ \mathbf{(1)}$$

On putting the values from Eq. (i) and (ii), we get

$$\text{Required vector} = \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \frac{8(2\hat{i} - 2\hat{j} - \hat{k})}{24}$$

$$= \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k} \quad \mathbf{(1)}$$

53. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$, then find

Given, $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$

Now, $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$

$$= 6\vec{a} \cdot \vec{a} + 21\vec{a} \cdot \vec{b} - 10\vec{b} \cdot \vec{a} - 35\vec{b} \cdot \vec{b}$$

$$= 6|\vec{a}|^2 + 21\vec{a} \cdot \vec{b} - 10\vec{a} \cdot \vec{b} - 35|\vec{b}|^2 \quad \mathbf{(1)}$$

$$[\because \vec{x} \cdot \vec{x} = |\vec{x}|^2 \text{ and } \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$

$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

$= 6(2)^2 + 11(1) - 35(1)^2 = 0$ **(1)**

$[\because |\vec{a}| = 2 \text{ and } |\vec{b}| = 1]$

Hence, $(3\vec{a} - 5\vec{b})\cdot(2\vec{a} + 7\vec{b}) = 0$ **(1)**

54. If vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Foreign 2011; All India 2009C

Given, $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$.

Also, $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$.

$\therefore \quad (\vec{a} + \lambda\vec{b})\cdot\vec{c} = 0$ ...(i)**(1)**

$[\because \text{when } \vec{a} \perp \vec{b}, \text{ then } \vec{a}\cdot\vec{b} = 0]$

Now, $\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

$\Rightarrow \vec{a} + \lambda\vec{b} = \hat{i}(2 - \lambda) + \hat{j}(2 + 2\lambda) + \hat{k}(3 + \lambda)$ **(1)**

Then, from Eq. (i), we get

$[\hat{i}(2 - \lambda) + \hat{j}(2 + 2\lambda)$

$+ \hat{k}(3 + \lambda)]\cdot[3\hat{i} + \hat{j}] = 0$ **(1)**

$\Rightarrow \quad 3(2 - \lambda) + 1(2 + 2\lambda) = 0$

$\Rightarrow \quad 8 - \lambda = 0$

$\therefore \quad \lambda = 8$ **(1)**

55. Using vectors, find the area of triangle with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$.

All India 2011

Do some as Que. 47. **[Ans.** $\frac{1}{2}\sqrt{61}$ sq units**]**

56. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors, such that $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$ and each one of these is perpendicular to the sum of other two, then find $|\vec{a} + \vec{b} + \vec{c}|$. All India 2011C, 2010C

Given, $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ ...(i)

Also, given that each of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is perpendicular to sum of the other two vectors, i.e.

$$\vec{a} \perp (\vec{b} + \vec{c})$$

$$\Rightarrow \quad \vec{a} \cdot (\vec{b} + \vec{c}) = 0$$

$$\Rightarrow \quad \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0 \quad \text{...(ii)}$$

$$\vec{b} \perp (\vec{c} + \vec{a})$$

$$\Rightarrow \quad \vec{b} \cdot (\vec{c} + \vec{a}) = 0$$

$$\Rightarrow \quad \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{a} = 0 \quad \text{...(iii)}$$

and $\vec{c} \perp (\vec{a} + \vec{b}) \Rightarrow \vec{c} \cdot (\vec{a} + \vec{b}) = 0$

$$\Rightarrow \quad \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0 \quad \text{...(iv)}$$

[$\because$ when two vectors are perpendicular, then their dot product is zero] **(1)**

Now, adding Eqs. (ii), (iii) and (iv), we get

$$2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=0 \quad \ldots(v)$$

$[\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}, \vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{b}$ and $\vec{c}\cdot\vec{a}=\vec{a}\cdot\vec{c}]$

Now, consider

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$$
$$+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}$$

$$\Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$$
$$+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}) \quad \textbf{(1)}$$

$[\because \vec{a}.\vec{a}=|\vec{a}|^2]$

On putting the values from Eqs. (i) and (v) we get

$$|\vec{a}+\vec{b}+\vec{c}|^2=(3)^2+(4)^2+(5)^2+2(0) \quad \textbf{(1)}$$
$$=9+16+25=50$$

Hence, $|\vec{a}+\vec{b}+\vec{c}|=\sqrt{50}$

$$=\sqrt{25\times 2}=5\sqrt{2} \quad \textbf{(1)}$$

57. Using vectors, find the area of triangle with vertices $A(2,3,5)$, $B(3,5,8)$ and $C(2,7,8)$.
Delhi 2010C

Do same as Que. 47. **[Ans.** $\frac{1}{2}\sqrt{61}$ sq units**]**

58. The scalar product of vector $\hat{i}+\hat{j}+\hat{k}$ with the unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.
All India 2009, 2008C

Do same as Que. 40. **[Ans.** $\lambda=1$**]**

59. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, then show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$. **HOTS; Delhi 2009**

> If two vectors are parallel, then their cross-product will be a zero vector.

Given, $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ ...(i)

and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$ ...(ii)**(1)**

On subtracting Eq. (ii) from Eq. (i), we get

$$(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) = (\vec{c} \times \vec{d}) - (\vec{b} \times \vec{d})$$

$$\Rightarrow (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{d}) - (\vec{c} \times \vec{d}) = \vec{0}$$

$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) + (\vec{b} - \vec{c}) \times \vec{d} = \vec{0}$$

$$\Rightarrow \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c}) = \vec{0}$$

$[\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}]$**(1)**

$$\Rightarrow (\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$$

$[\because \vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}]$ **(1/2)**

Thus, we have that cross-product of vectors $\vec{a} - \vec{d}$ and $\vec{b} - \vec{c}$ as a zero vector, so $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$. **(1½)**

60. Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a} + \vec{b} + \vec{c} = 0$. Find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$, if $|\vec{a}| = 1, |\vec{b}| = 4$ and $|\vec{c}| = 2$. **All India 2008C**

Do same as Que. 50. $\left[\textbf{Ans.} -\frac{21}{2}\right]$

61. Find a vector of magnitude 5 units, perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$. **Delhi 2008C**

Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \quad \vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 2\hat{i} + 3\hat{j} + 4\hat{k}$ **(1)**

and $\quad \vec{a} - \vec{b} = \hat{i} + \hat{j} + \hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= -\hat{j} - 2\hat{k}$

Let $\quad \vec{a} + \vec{b} = \vec{c}$ and $\vec{a} - \vec{b} = \vec{d}$

Then we get, $\vec{c} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

and $\quad \vec{d} = -\hat{j} - 2\hat{k}$

We know that, unit vector which is perpendicular to both $\vec{c}$ and $\vec{d}$ is given by

$$\hat{n} = \frac{\vec{c} \times \vec{d}}{|\vec{c} \times \vec{d}|}$$

$$\therefore \quad \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$= \hat{i}(-6 + 4) - \hat{j}(-4 - 0) + \hat{k}(-2 - 0)$ **(1)**

$= -2\hat{i} + 4\hat{j} - 2\hat{k}$

and $|\vec{c} \times \vec{d}| = \sqrt{(-2)^2 + (4)^2 + (-2)^2}$

$= \sqrt{4 + 16 + 4}$

$= \sqrt{24} = 2\sqrt{6}$

$$= \sqrt{24} = 2\sqrt{6}$$

$$\therefore \qquad \hat{n} = \frac{-2\hat{i} + 4\hat{j} - 2\hat{k}}{2\sqrt{6}}$$

$$= \frac{-\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}} \qquad \textbf{(1)}$$

Hence, required vector of magnitude 5 units

$$= 5\left(\frac{-\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}}\right)$$

$$= -\frac{5}{\sqrt{6}}\hat{i} + \frac{10}{\sqrt{6}}\hat{j} - \frac{5}{\sqrt{6}}\hat{k} \qquad \textbf{(1)}$$