UNIT-IV

VECTORS & THREE-DIMENSIONAL GEOMETRY

CHAPTER 10 VECTORS

Syllabus

Vectors and Scalars, Magnitude and direction of a vector. Direction cosines and direction ratios of a vector , "types of vectors" equal, unit, zero, parallel and collinear vectors, position vector of a "point, negative of a vector", components of a vector, addition of vectors (properties of addition, laws of addition), Multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical interpretation, properties and application of Scalar (dot) product of vectors, Vector (cross) product of vectors, scalar triple product of vectors.

Chapter Analysis

TOPIC	2016		2017		2018
	Delhi	OD	Delhi	OD	Delhi/OD
Properties	2 Q. (1 Mark)	–	–	–	–
Angle between vectors	–	–	1 Q. (4 Marks)	–	2 Q. (1 Mark) 2 Q. (2 Marks)
Dot Product	–	–	–	–	–
Cross product	–	2 Q. (1 Mark)	–	–	1 Q. (4 Marks)
Area of triangle	–	1 Q. (4 Marks)	–	1 Q. (4 Marks)	
Coplanarity	1 Q. (4 Marks)	–	1 Q. (4 Marks)	1 Q. (4 Marks)	–

TOPIC-1
Basic Algebra of Vectors

Revision Notes

1. Vector : Basic Introduction :

- A quantity having magnitude as well as the direction is called a vector. It is denoted as $\vec{AB}$ or $\vec{a}$. Its magnitude (or modulus) is $|\vec{AB}|$ or $|\vec{a}|$ otherwise, simply AB or a.
- Vectors are denoted by symbols such as $\vec{a}$. [Pictorial representation of vector]

2. Initial and Terminal Points :

The initial and terminal points means that point from which the vector originates and terminates respectively.

3. Position Vector :

The position vector of a point say $P(x, y, z)$ is $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and the magnitude is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

The vector $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is said to be in its **component form**. Here x, y, z are called the scalar components or rectangular components of $\vec{r}$ and $x\hat{i}, y\hat{j}, z\hat{k}$ are the vector components of $\vec{r}$ along x, y, z-axis respectively.

- Also, $\vec{AB}$ = (Position Vector of B) – (Position Vector of A). For example, let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Then, $\vec{AB} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$.
- Here $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors along the axes OX, OY and OZ respectively (The discussion about unit vectors is given later under 'types of vectors').

4. Direction Ratios and Direction Cosines :

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then coefficient of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{r}$ *i.e.*, x, y, z are called the direction ratios (abbreviated as d.r.'s) of vector $\vec{r}$. These are denoted by a, b, c (*i.e.*, $a = x$, $b = y$, $c = z$; in a manner we can say that scalar components of vector $\vec{r}$ and its d.r.'s both are the same).

Also, the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{r}$ (which is the unit vector of $\vec{r}$) *i.e.*, $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$ are called direction cosines (which is abbreviated as d.c.'s) of vector $\vec{r}$.

- These direction cosines are denoted by l, m, n such that $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ and $l^2 + m^2 + n^2 = 1$ $\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- It can be easily concluded that $\frac{x}{r} = l = \cos\alpha$, $\frac{y}{r} = m = \cos\beta$, $\frac{z}{r} = n = \cos\gamma$.

  Therefore, $\vec{r} = lr\hat{i} + mr\hat{j} + nr\hat{k} = r(\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k})$. [Here $r = |\vec{r}|$].

5. Addition of vectors

(a) Triangular law : If two adjacent sides (say sides AB and BC) of a triangle ABC are represented by $\vec{a}$ and $\vec{b}$ taken in same order, then the third side of the triangle taken in the reverse order gives the sum of vectors $\vec{a}$ and $\vec{b}$ *i.e.*, $\vec{AC} = \vec{AB} + \vec{BC} \Rightarrow \vec{AC} = \vec{a} + \vec{b}$.

- Also since $\vec{AC} = -\vec{CA} \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$.
- And $\vec{AB} + \vec{BC} - \vec{AC} = 0 \Rightarrow \vec{AB} + \vec{BC} + \vec{CA} = 0$.

(b) Parallelogram law : If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and the direction by the two adjacent sides (say AB and AD) of a parallelogram $ABCD$, then their sum is given by that diagonal of parallelogram which is co-initial with $\vec{a}$ and $\vec{b}$ *i.e.*, $\vec{OC} = \vec{OA} + \vec{OB}$.

6. Properties of Vector Addition

(a) Commutative property : $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ be any two given vectors,

then $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k} = \vec{b} + \vec{a}$.

(b) Associative property : $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

(c) Additive identity property : $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.

(d) Additive inverse property : $\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$.

Note : Multiplication of a vector by a scalar

Let $\vec{a}$ be any vector and k be any scalar. Then the product $k\vec{a}$ is defined as a vector whose magnitude is $|k|$ times that of $\vec{a}$ and the direction is

(i) same as that of $\vec{a}$ if k is positive, and **(ii)** opposite as that of $\vec{a}$ if k is negative.

Know the Terms

Types of Vectors :

(a) Zero or Null vector : It is that vector whose initial and terminal points are coincident. It is denoted by $\vec{0}$. Ofcourse its magnitude is 0 (zero).

- Any non-zero vector is called a **proper vector**.

(b) Co-initial vectors : Those vectors (two or more) having the same starting point are called the co-initial vectors.

(c) Co-terminus vectors : Those vectors (two or more) having the same terminal point are called the co-terminus vectors.

(d) Negative of a vector : The vector which has the same magnitude as the $\vec{r}$ but opposite direction. It is denoted by $-\vec{r}$. Hence if, $\vec{AB} = \vec{r}$ or $\vec{BA} = -\vec{r}$ *i.e.,* $\vec{AB} = -\vec{BA}$, $\vec{PQ} = -\vec{QP}$ etc.

(e) Unit vector : It is a vector with the unit magnitude. The unit vector in the direction of vector $\vec{r}$ is given by $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$ such that $|\hat{r}| = 1$, so, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then its unit vector is :

$$\hat{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}.$$

- Unit vector perpendicular to the plane $\vec{a}$ and $\vec{b}$ is : $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$.

(f) Reciprocal of a vector : It is a vector which has the same direction as the vector $\vec{r}$ but magnitude equal to the reciprocal of the magnitude of $\vec{r}$. It is denoted as $\vec{r}^{\,-1}$. Hence $\left|\vec{r}^{\,-1}\right| = \frac{1}{|\vec{r}|}$.

(g) Equal vectors : Two vectors are said to be equal if they have the same magnitude as well as direction, regardless of the position of their initial points.

Thus $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} |\vec{a}| = |\vec{b}| \\ \vec{a} \text{ and } \vec{b} \text{ have same direction} \end{cases}$

Also, if $\vec{a} = \vec{b} \Rightarrow a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = b_1\hat{i} + b_2\hat{j} + a_3\hat{k} \Rightarrow a_1 = b_1,\ a_2 = b_2, a_3 = b_3$.

(h) Collinear or Parallel vector : Two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if there exists a non-zero scalar λ such that $\vec{a} = \lambda\vec{b}$.

- It is important to note that the respective coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\vec{a}$ and $\vec{b}$ are proportional provided they are parallel or collinear to each other.
- The d.r's of parallel vectors are same (or are in proportion).
- The vectors $\vec{a}$ and $\vec{b}$ will have same or opposite direction as λ is positive or negative respectively.
- The vectors $\vec{a}$ and $\vec{b}$ are collinear if $\vec{a}\times\vec{b} = \vec{0}$.

(i) Free vectors : The vectors which can undergo parallel displacement without changing its magnitude and direction are called free vectors.

Know the Formulae

The position vector of a point say P dividing a line segment joining the points A and B whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio $m : n$.

(a) Internally, $\vec{OP} = \frac{m\vec{b} + n\vec{a}}{m+n}$ **(b)** Externally, $\vec{OP} = \frac{m\vec{b} - n\vec{a}}{m-n}$

- Also if point P is the mid-point of line segment AB, then $\vec{OP} = \frac{\vec{a}+\vec{b}}{2}$.

Objective Type Questions (1 mark each)

Q. 1. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$, respectively is

(a) $\frac{1}{2}$ **(b) 1**

(c) 2 **(d) 4**

[NCERT Ex.]

Ans. Correct option : (c)

Explanation :

The position vectors of vertices A, B, C and D of rectangle ABCD are given as :

$\overrightarrow{OA}=-\hat{i}+-\hat{j}+\hat{k}$, $\overrightarrow{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$,

$\overrightarrow{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$, $\overrightarrow{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as :

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}$$

$$=\hat{k}(-2)=-2\hat{k}$$

$$\Rightarrow \left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$$

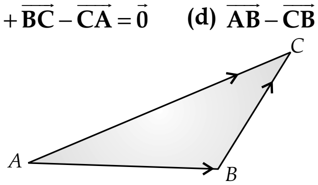
Now, it is known that the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

So that, the area of the given rectangle is $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$ sq. units.

Q. 2. In triangle ABC (Figure), which of the following is not true :

(a) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$ **(b)** $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

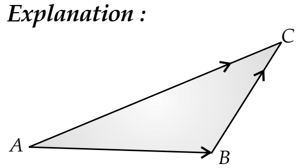
(c) $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$ **(d)** $\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$



[NCERT Ex.]

Ans. Correct option : (c)

Explanation :



Applying the triangle law of addition in the above triangle, we have

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ ...(i)

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}=-\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=0$ ...(ii)

$\therefore$ The equation given in alternative (a) is true.

$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

$\therefore$ The equation given in alternative (b) is true.

From equation (ii), we have

$\overrightarrow{AB}-\overrightarrow{CB}+\overrightarrow{CA}=\vec{0}$

The equation given in alternative (d) is true.

Now, consider the equation given in alternative (c) :

$\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{CA}=\vec{0}$

$\Rightarrow \quad \overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{CA}$ ...(iii)

For equations (i) and (iii), we have :

$\overrightarrow{AC}=\overrightarrow{CA}$

$\Rightarrow \overrightarrow{AC}=-\overrightarrow{AC}$

$\Rightarrow \overrightarrow{AC}+\overrightarrow{AC}=\vec{0}$

$\Rightarrow \quad 2\overrightarrow{AC}=\vec{0}$

$\Rightarrow \overrightarrow{AC}=\vec{0}$, which is not true.

So that, the equation given in alternative (c) is incorrect.

Q. 3. If $\vec{a}$ is a non-zero vector of magnitude '*a*' and λ a non-zero scalar, then $\lambda\vec{a}$ is unit vector if

(a) $\lambda = 1$ **(b)** $\lambda = -1$

(c) $a = |\lambda|$ **(d)** $a = 1/|\lambda|$

[NCERT Ex.]

Ans. Correct option : (d)

Explanation :

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}|=1$.

Now,

$$|\lambda\vec{a}|=1$$

$$\Rightarrow |\lambda||\vec{a}|=1$$

$$\Rightarrow \quad |\vec{a}|=\frac{1}{|\lambda|} \qquad [\lambda\neq 0]$$

$$\Rightarrow \quad a=\frac{1}{|\lambda|} \qquad [|\vec{a}|=a]$$

So that, vector $\lambda\vec{a}$ is a unit vector if $a=\frac{1}{|\lambda|}$.

Q. 4. If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then which of the following are incorrect :

(a) $\vec{b}=\lambda\vec{a}$**, for some scalar λ**

(b) $\vec{a}=\pm\vec{b}$

(c) the respective components of $\vec{a}$ and $\vec{b}$ are not proportional

(d) both the vectors $\vec{a}$ and $\vec{b}$ have same direction, but different magnitudes

[NCERT Ex.]

Ans. Correct option : (d)

Explanation :
If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then they are parallel.
Therefore, we have

$\vec{b} = \lambda\vec{a}$ (For some scalar λ)

If $\lambda = \pm 1$, then $\vec{a} = \pm\vec{b}$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then,

$$\vec{b} = \lambda\vec{a}$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$$

$$\Rightarrow b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

$$\Rightarrow \quad b_1 = \lambda a_1, b_2 = \lambda a_2, b_3 = \lambda a_3$$

$$\Rightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

So that, the respective components of $\vec{a}$ and $\vec{b}$ are proportional. However, vectors $\vec{a}$ and $\vec{b}$ can have different directions. Hence, the statement given in option (d) is incorrect.

Q. 5. The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

(a) $\hat{i} - 2\hat{j} + 2\hat{k}$ (b) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

(c) $3(\hat{i} - 2\hat{j} + 2\hat{k})$ (d) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

[NCERT Exemp.]

Ans. Correct option : (c)
Explanation :
Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Any vector in the direction of a vector $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$.

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

∴ Vector in the direction of $\vec{a}$ with magnitude 9.

$$= 9\ \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$$= 3(\hat{i} - 2\hat{j} + 2\hat{k})$$

Q. 6. The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$ in the ratio 3 :1 is :

(a) $\frac{3\vec{a} - 2\vec{b}}{2}$ (b) $\frac{7\vec{a} - 8\vec{b}}{4}$

(c) $\frac{3\vec{a}}{4}$ (d) $\frac{5\vec{a}}{4}$

[NCERT Exemp.]

Ans. Correct option : (d)
Explanation :
Let the position vector of the R divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a} + \vec{b}$.

$$\therefore \text{Position vector, } R = \frac{3(\vec{a} + \vec{b}) + 1(2\vec{a} - 3\vec{b})}{3 + 1}$$

Since, the position vector of a point R dividing the line segments joining the points P and Q, whose position vectors are $\vec{p}$ and $\vec{q}$ in the ration $m : n$ internally, is given by $\frac{m\vec{q} + n\vec{p}}{m + n}$.

$$\therefore R = \frac{5\vec{a}}{4}.$$

Very Short Answer Type Questions (1 mark each)

Q. 1. Find a vector in the direction of $\vec{a} = \hat{i} - 2\hat{j}$ that has magnitude 7 units. R&U **[NCERT]**
[Delhi Set I, II, III Comptt. 2015]

Sol.
$$\hat{a} = \frac{1}{\sqrt{5}}\hat{i} - \frac{2}{\sqrt{5}}\hat{j} = \frac{\vec{a}}{|\vec{a}|} \qquad ½$$

then
$$7\hat{a} = \frac{7}{\sqrt{5}}\hat{i} - \frac{14}{\sqrt{5}}\hat{j} \qquad ½$$

[CBSE Marking Scheme 2015]

Q. 2. Write a vector in the direction of the vector $\hat{i} - 2\hat{i} + 2\hat{k}$ that has magnitude 9 units.
R&U **[Delhi Set I Comptt. 2014]**

Sol. Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

The vector in the direction of $\vec{a}$ with magnitude 9 is $9 \times \hat{a}$.

$$\therefore \quad \text{Required vector} = 9 \times \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + (-2)^2 + 2^2}}$$

$$= 9 \times \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$$= 3\hat{i} - 6\hat{j} + 6\hat{k} \qquad 1$$

Q. 3. Write a unit vector in the direction of the sum of vectors $\vec{a} = 2\vec{i} + 2\vec{j} - 5\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} - 7\vec{k}$.
R&U **[Delhi Set III, 2014]**

Sol. Let $\vec{r} = \vec{a} + \vec{b} = 4\vec{i} + 3\vec{j} - 12\vec{k}$

$$|\vec{r}| = \sqrt{16 + 9 + 144} = \sqrt{169} = 13 \qquad ½$$

So, unit vector

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13}$$

$$= \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k} \qquad ½$$

Q. 4. Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude 21 units.
R&U **[Foreign Set I, II, III, 2014]**

Sol. Let $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

The vector in the direction of $\vec{a}$ with a magnitude 21 is 21 × $\hat{a}$.

∴ Required vector = $21 \times \frac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{2^2+(-3)^2+6^2}}$

$= 21 \times \frac{2\hat{i}-3\hat{j}+6\hat{k}}{7}$

$= 6\hat{i}-9\hat{j}+18\hat{k}$ **1**

Q. 5. If $\vec{a} = x\hat{i}+2\hat{j}-z\hat{k}$ and $\vec{b} = 3\hat{i}-y\hat{j}+\hat{k}$ are two equal vectors, then write the value of $x + y + z$. R&U **[Delhi Set I, 2013]**

Sol. $\vec{a} = x\hat{i}+2\hat{j}-z\hat{k}$

and $\vec{b} = 3\hat{i}-y\hat{j}+\hat{k}$

are equal vectors

So, $\vec{a} = \vec{b}$

or $x\hat{i}+2\hat{j}-z\hat{k} = 3\hat{i}-y\hat{j}+\hat{k}$ ½

∴ $x = 3, y = -2, z = -1$

∴ $x + y + z = 3 - 2 - 1 = 0.$ ½

Q. 6. Write a unit vector in the direction of the sum of vectors :

$\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ **and** $\vec{b} - \hat{i}+\hat{j}+3\hat{k}$

R&U **[NCERT] [Delhi Set III, 2013]**

Sol. Given, $\vec{a} = 2\hat{i}-j+2\hat{k}$

and $\vec{b} = -\hat{i}+j+3\hat{k}$

Let, $\vec{r} = \vec{a}+\vec{b}$

$= (2\hat{i}-\hat{j}+2\hat{k})+(-\hat{i}+\hat{j}+3\hat{k})$

$= \hat{i}+5\hat{k}$

$|\vec{r}| = \sqrt{(1)^2+(5)^2} = \sqrt{26}$ ½

So, required unit vector

$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i}+5\hat{k}}{\sqrt{26}}$

$= \frac{1}{\sqrt{26}}\hat{i}+\frac{5}{\sqrt{26}}\hat{k}$ ½

Commonly Made Error

- Generally students commit errors in finding the unit vector as they don't get the result in required vector form.

Q. 7. Find a unit vector parallel to the sum of vectors $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-3\hat{j}+5\hat{k}$.

R&U **[Delhi Set I Comptt. 2012]**

Sol. Sum of given two vectors is given as

$(\hat{i}+\hat{j}+\hat{k})+(2\hat{i}-3\hat{j}+5\hat{k})$

$= (1+2)\hat{i}+(1-3)\hat{j}+(1+5)\hat{k}$

$= 3\hat{i}-2\hat{j}+6\hat{k} = \vec{A}$ (say) ½

A unit vector parallel to this vector

$= \frac{3\hat{i}-2\hat{j}+6\hat{k}}{|\vec{A}|}$

$= \frac{3\hat{i}-2\hat{j}+6\hat{k}}{\sqrt{3^2+(-2)^2+6^2}} = \frac{3\hat{i}-2\hat{j}+6\hat{k}}{\sqrt{49}}$

$= \frac{3}{7}\hat{i}-\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}.$ ½

Q. 8. Find a unit vector in the direction of

$\vec{A} = 3\hat{i}-2\hat{j}+6\hat{k}.$

R&U **[O.D. Set I Comptt. 2012]**

Sol. A unit vector in the direction of vector $\vec{A}$ is given by

$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$

$|\vec{A}| = \sqrt{3^2+(-2)^2+6^2} = 7$

$\hat{A} = \frac{\vec{A}}{|A|} = \frac{3\hat{i}-2\hat{j}+6\hat{k}}{7}$

A unit vector in the direction of $\vec{A}$

$= \frac{3}{7}\hat{i}-\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}.$ **1**

Q. 9. Find the sum of the vectors $\vec{a} = \hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = -2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c} = \hat{i}-6\hat{j}-7\hat{k}$.

R&U **[NCERT] [Delhi Set I, 2012]**

Sol. $\vec{a}+\vec{b}+\vec{c}$

$= (\hat{i}-2\hat{j}+\hat{k})+(-2\hat{i}+4\hat{j}+5\hat{k})+(\hat{i}-6\hat{j}-7\hat{k})$

$= (1-2+1)\hat{i} + (-2+4-6)\hat{j} + (1+5-7)\hat{k}$

$= 0\hat{i} - 4\hat{j} - 1\hat{k} = -4\hat{j} - \hat{k}.$ **1**

Q.10. Find the sum of the vectors :

$\vec{a} = \hat{i}-2\hat{j}, \vec{b} = 2\hat{i}-3\hat{j}, \vec{c} = 2\hat{i}+3\hat{k}.$

R&U **[Delhi Set II, 2012]**

Sol. $\vec{a}+\vec{b}+\vec{c} = (\hat{i}-2\hat{j})+(2\hat{i}-3\hat{j})+(2\hat{i}+3\hat{k})$

$= (1+2+2)\hat{i}+(-2-3)\hat{j}+3\hat{k}$

$= 5\hat{i}-5\hat{j}+3\hat{k}.$ **1**

Q. 11. Find the sum of the vectors :

$$\vec{a} = \hat{i} - 3\hat{k}, \vec{b} = 2\hat{j} - \hat{k}, \vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}.$$

R&U [Delhi Set III, 2012]

Sol.
$$\vec{a} + \vec{b} + \vec{c} = (\hat{i} - 3\hat{k}) + (2\hat{j} - \hat{k}) + (2\hat{i} - 3\hat{j} + 2\hat{k})$$
$$= (1+2)\hat{i} + (2-3)\hat{j} + (-3-1+2)\hat{k}$$
$$= 3\hat{i} - \hat{j} - 2\hat{k}. \quad 1$$

Q. 12. Write the number of vectors of unit length perpendicular to both the vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. **A [O.D. Set I, II, III 2016]**

Sol. There are two such vectors of unit length perpendicular to both the given vectors $\vec{a}$ and $\vec{b}$ and vectors are $\pm \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$.

Q. 13. Find the position vector of a point which divides the join of points with position vectors $(\vec{a} - 2\vec{b})$ and $(2\vec{a} + \vec{b})$ externally in the ratio 2 : 1. **R&U [Delhi Set I, II, III 2016]**

Sol. Required vector $= \dfrac{1(\vec{a} - 2\vec{b}) - 2(2\vec{a} + \vec{b})}{1-2}$ ½

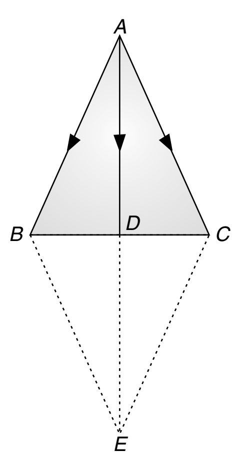
$$= \frac{(\vec{a} - 2\vec{b}) - (4\vec{a} + 2\vec{b})}{-1}$$

$$= 3\vec{a} + 4\vec{b} \quad ½$$

AI Q. 14. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides *AB* and *AC*, respectively of ΔABC. Find the length of the median through *A*.

A [Delhi Set I, II, III 2016] [Foreign 2015]

Sol. $\overrightarrow{AB} = \hat{j} + \hat{k}$ and $\overrightarrow{AC} = 3\hat{j} - \hat{j} + 4\hat{k}$



Now *ABEC* represent a parallelogram with *AE* as the diagonal.

$$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{AC} \quad ½$$
$$= (\hat{j} + \hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k}) = 3\hat{i} + 5\hat{k}$$

Now, $|\overrightarrow{AE}| = \sqrt{(3)^2 + (5)^2} = \sqrt{9+25} = \sqrt{34}$

$\therefore \quad |\overrightarrow{AD}| = \dfrac{1}{2}\sqrt{34}$ units ½

Q. 15. Write the position vector of the point which divides the join of points with position vectors $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ in the ratio 2:1. **R&U**

Sol. Let
$$\overrightarrow{OP} = 3\vec{a} - 2\vec{b}$$
$$\overrightarrow{OQ} = 2\vec{a} + 3\vec{b}$$

The position vector of the point *R* dividing the join of *P* and *Q* internally in the ratio 2 : 1 is

$$\overrightarrow{OR} = \frac{2(2\vec{a} + 3\vec{b}) + (3\vec{a} - 2\vec{b})}{2+1} \quad ½$$
$$= \frac{4\vec{a} + 6\vec{b} + 3\vec{a} - 2\vec{b}}{3}$$
$$= \frac{7\vec{a}}{3} + \frac{4\vec{b}}{3} \quad ½$$

Q. 16. Write the value of $\vec{p}$ for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel vectors. **A [O.D. Set I, 2014]**

Sol.
$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \quad \text{[For parallel vectors]}$$

or
$$\frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$

or
$$p = -\frac{1}{3} \quad 1$$

Q. 17. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$ making an angle of $\dfrac{\pi}{4}$ with *x*-axis, $\dfrac{\pi}{2}$ with *y*-axis and an acute angle θ with *z*-axis. **A [O.D. Set II 2014]**

Sol. Let $\hat{a}$ be the unit vector in the direction of vector $\vec{a}$.

Since vector $\vec{a}$ makes an angle of $\dfrac{\pi}{4}$ with *x*-axis, $\dfrac{\pi}{2}$ with *y*-axis and an acute angle θ with *z*-axis therefore

$$\cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{2} + \cos^2\theta = 1$$

[Using $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$]

or $\cos^2\theta = \frac{1}{2}$

or $\cos\theta = \frac{1}{\sqrt{2}}$

or $\theta = \frac{\pi}{4}$

Therefore, $\vec{a} = 5\sqrt{2}\,\hat{a}$

$= 5\sqrt{2}\left(\cos\frac{\pi}{4}\hat{i} + \cos\frac{\pi}{2}\hat{j} + \cos\frac{\pi}{4}\hat{k}\right)$

$= 5\hat{i} + 5\hat{k}$ **1**

Q. 18. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ. **A [Delhi Set I, 2013]**

Sol. We know that if a vector $\vec{a}$ makes angle α, β & γ with $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively, then

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Here, we have

$\alpha = \frac{\pi}{3}$, $\beta = \frac{\pi}{4}$ and $\gamma = \theta$, an acute angle

$\therefore \cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{4} + \cos^2\theta = 1$ ½

or $\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$

or $\cos^2\theta = \frac{1}{4}$

or $\cos\theta = \pm\frac{1}{2}$ or $\theta = \frac{\pi}{3}$. ½

Q. 19. *P* and *Q* are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point *R* which divides the line segment *PQ* externally in the ratio 2 : 1.

R&U [NCERT] [O.D. Set I, 2013]

Sol. Consider two points *P* and *Q* with position vectors $\vec{OP} = 3\vec{a} - 2\vec{b}$ and $\vec{OQ} = \vec{a} + \vec{b}$, then position vector of the point *R* dividing the join of *P* and *Q* externally in the ratio 2 : 1 is

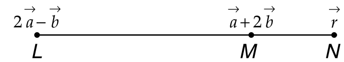
$$\vec{OR} = \frac{2(\vec{a} + \vec{b}) - (3\vec{a} - 2\vec{b})}{2-1}$$

$= 4\vec{b} - \vec{a}$. **1**

Q. 20. *L* and *M* are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$ respectively. Write the position vectors of a point *N* which divides the line segment *LM* in the ratio 2:1 externally.

R&U [O.D. Set I, 2013]

Sol. If $\vec{r}$ is the position vector of *N*, then by section formula



$$\vec{r} = \frac{2(\vec{a} + 2\vec{b}) - 1(2\vec{a} - \vec{b})}{2-1}$$

$$= \frac{2\vec{a} + 4\vec{b} - 2\vec{a} + \vec{b}}{1}$$

$= 5\vec{b}$ **1**

Q.21. Find the scalar components of the vector $\vec{AB}$ with initial point *A* (2, 1) and terminal point *B* (– 5, 7).

R&U [O.D. Set I, II, III, 2012]

Sol. $\vec{AB}$ = Position vector of *B* – Position vector of *A*

$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + 1\hat{j})$

$= (-5 - 2)\hat{i} + (7 - 1)\hat{j}$

$= -7\hat{i} + 6\hat{j}$ **1**

∴ The scalar components are (– 7, 6).

Q. 22. If a line has direction ratios 2, – 1, – 2, then what are its direction cosines ?

A [Delhi Set I, II, III, 2012]

Sol. Here direction ratios of line are 2, – 1, – 2.

∴ Direction cosines of line are $\frac{2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$, $\frac{-1}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$, $\frac{-2}{\sqrt{2^2 + (-1)^2 + (-2)^2}}$

i.e., $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$.

[**Note :** If *a*, *b*, *c* are the direction ratios of a line, the direction cosines are $\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\frac{b}{\sqrt{a^2+b^2+c^2}}$, $\frac{c}{\sqrt{a^2+b^2+c^2}}$] **1**

Q. 23. If $\vec{a}$ and $\vec{b}$ denote the position vectors of points *A* and *B* respectively and *C* is a point on *AB* such that *AC* = 2 *CB*, then write the position vector of *C*. **R&U [Outside Delhi Set I, II, III comptt. 2016]**

Sol. *AC* : *CB* = 2 :1

Position vector of C

$= \frac{\vec{a} + 2\vec{b}}{3}$ **1**

[CBSE Marking Scheme 2016]

Q. 24. If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector parallel to the vector $\vec{a} + \vec{b}$.

R&U [Outside Delhi Set I, II, III comptt. 2016]

Sol. $\vec{a}+\vec{b} = 6\hat{i}-3\hat{j}+2\hat{k}$ ½

Unit vector parallel to $\vec{a}+\vec{b}$ is $\frac{1}{7}(6\hat{i}-3\hat{j}+2\hat{k})$ ½

[CBSE Marking Scheme 2016]

Q. 25. Give an example of vectors $\vec{a}$ and $\vec{b}$ such that $|\vec{b}| = |\vec{b}|$ but $\vec{a} \neq \vec{b}$. R&U **[SQP 2017-18]**

Sol. $\vec{a} = \hat{i}, \vec{b} = \hat{j}$ [or any other correct answer] **1**

[CBSE Marking Scheme, 2017-18]

Q. 26. Write a unit vector in the direction of vector $\vec{PQ}$ where $\vec{P}$ and $\vec{Q}$ are the points (1, 3, 0) and (4, 5, 6), respectively, R&U **[Foreign 2014]**

Sol. Given points are $\vec{P}$ (1, 3, 0) and $\vec{Q}$ (4, 5, 6).

Here, $x_1 = 1, y_1 = 3, z_1 = 0$

and $x_2 = 4, y_2 = 5, z_2 = 6$

So, vector $PQ = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

$= (4-1)\hat{i} + (5-3)\hat{j} + (6-0)\hat{k}$

$= 3\hat{i}+2\hat{j}+6\hat{k}$ ½

∴ Magnitude of given vector

$|\vec{PQ}| = \sqrt{3^2+2^2+6^2} = \sqrt{9+4+36}$

$= \sqrt{49} = 7$ units

Hence, the unit vector in the direction of $\vec{PQ}$ is

$\frac{\vec{PQ}}{|\vec{PQ}|} = \frac{3\hat{i}+2\hat{j}+6\hat{k}}{7} = \frac{3}{7}\hat{i}+\frac{2}{7}\hat{j}+\frac{6}{7}\hat{k}.$ ½

Q. 27. For what values of $\vec{a}$, the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $a\hat{i}+6\hat{j}-8\hat{k}$ are collinear ?

R&U **[HOTS; Delhi 2011]**

Sol. Let given vectors are $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$ and

$\vec{b} = a\hat{i}+6\hat{j}-8\hat{k}$

vectors $\vec{a}$ and $\vec{b}$ are said to be collinear, if

$\vec{a} = k.\vec{b}$, where k is a scalar.

∴ $2\hat{i}-3\hat{j}+4\hat{k} = k(a\hat{i}+6\hat{j}-8\hat{k})$

On comparing the coefficients of $\hat{i}$ and $\hat{j}$, we get

$2 = ka$ and $-3 = 6k$ or $k = -\frac{1}{2}$

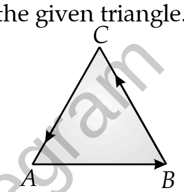
∴ $2 = -\frac{1}{2}a$ or $a = -4$ **1**

Answering Tips

- Clarify the concept of collinearity of two vectors.

Q. 28. If A, B and C are the vertices of a ΔABC, then what is the value of $\vec{AB}+\vec{BC}+\vec{CA}$? R&U **[Delhi 2011C]**

Sol. Let ΔABC be the given triangle.



Now, by triangle law of vector addition, we have $\vec{AB}+\vec{BC} = \vec{AC}$

or $\vec{AB}+\vec{BC}+\vec{CA} = \vec{CA}+\vec{AC}$

[adding $\vec{CA}$ on both sides]

or $\vec{AB}+\vec{BC}+\vec{CA} = \vec{CA}-\vec{CA}$ [$\because \vec{AC} = -\vec{CA}$]

∴ $\vec{AB}+\vec{BC}+\vec{CA} = \vec{0}$ **1**

Q. 29. Find the unit vector in the direction of the sum of vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $4\hat{i}-3\hat{j}-2\hat{k}$.

R&U **[Foreign 2015]**

Sol. Try Yourself

Q. 30. A and B are two points with position vectors $2\vec{a}-3\vec{b}$ and $6\vec{b}-\vec{a}$ respectively. Write the position vector of a point P which divides the line segment AB internally in the ratio 1 : 2.

R&U **[All India 2013]**

Sol. Try Yourself

Q. 31. Write the position vector of mid-point of the vector joining points P(2, 3, 4) and Q(4, 1, – 2).

R&U **[Foreign 2011]**

Sol. Try Yourself

Q. 32. Write a unit vector in the direction of vector $\vec{a} = 2\hat{i}+\hat{j}+2\hat{k}$ R&U **[All India 2011; Delhi 2009]**

Sol. Try Yourself

Q. 33. Find the magnitude of the vector $\vec{a} = 3\hat{i}+2\hat{j}+6\hat{k}$.

R&U **[All India 2011C; Delhi 2008]**

Sol. Try Yourself

Q. 34. Find a unit vector in the direction of vector $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$ R&U **[All India 2011C]**

Sol. Try Yourself

Q. 35. Find a unit vector in the direction of vector $\vec{a} = 2\hat{i}-3\hat{j}+6\hat{k}$ R&U **[Delhi 2011C]**

Sol. Try Yourself

Q. 36. Find a vector in the direction of $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$**, which has magnitude 6 units.** R&U **[Delhi 2010C]**

Sol. Try Yourself

Q. 37. Find a unit vector in the direction of vector $\vec{b} = 6\hat{i}-2\hat{j}+3\hat{k}$. R&U **[All India 2009C]**

Sol. Try Yourself

Q. 38. Find a unit vector in the direction of vector $\vec{a} = 2\hat{i}-6\hat{j}+3\hat{k}$. R&U **[Delhi 2009]**

Sol. Try Yourself

Q. 39. Find the position vector of mid-point of the line segment *AB*, where *A* is point (3, 4, – 2) and *B* is point (1, 2, 4). R&U **[Delhi 2010]**

Sol. Try Yourself

Q. 40. Write a vector of magnitude 9 units in the direction of vector $-2\hat{i}+\hat{j}+2\hat{k}$. R&U **[All India 2010]**

Sol. Try Yourself

Q. 41. Write a vector of magnitude 15 units in the direction of vector $\hat{i}-2\hat{j}+2\hat{k}$. R&U **[Delhi 2010]**

Sol. Try yourself

Short Answer Type Questions

(2 marks each)

Q. 1. Find the area of the parallelogram whose diagonals are represented by the vectors $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}$ **and** $\vec{b} = 2\hat{i}-\hat{j}+2\hat{k}$ R&U **[SQP 2018-19]**

Sol.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= -2\hat{i}+4\hat{j}+4\hat{k} \qquad 1$$

$$|\vec{a}\times\vec{b}| = \sqrt{4+16+16} = 6 \qquad ½$$

$$\text{Area of the parallelogram} = \frac{|\vec{a}\times\vec{b}|}{2} = 3\text{sq units.} \; ½$$

[CBSE Marking Scheme 2018-19]

Answering Tip

- Clarify the concept of finding area of parallelogram whose diagonal all vectors.

Q. 2. Find the angle between the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ **and** $\vec{b} = \hat{i}-\hat{j}+\hat{k}$ A **[SQP 2018-19]**

Sol. The angle θ between the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \qquad 1$$

$$i.e., \cos\theta = \frac{\left(\hat{i}+\hat{j}-\hat{k}\right).\left(\hat{i}-\hat{j}+\hat{k}\right)}{\sqrt{(1)^2+(1)^2+(-1)^2}.\sqrt{(1)^2+(-1)^2+(1)^2}}$$

$$i.e., \cos\theta = \frac{1-1-1}{\sqrt{3}.\sqrt{3}}$$

$$i.e., \cos\theta = -\frac{1}{3}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{3}\right). \qquad 1$$

[CBSE Marking Scheme 2018-19]

Answering Tip

- Concept of Angle between two vectors should be revised thoroughly.

Q. 3. Show that each of three vectors is a unit vector :

$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}), \frac{1}{7}(6\hat{i}-3\hat{j}+2\hat{k}), \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}).$$

R&U

Sol.

$$\vec{a} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\vec{b} = \frac{1}{7}(6\hat{i}-3\hat{j}+2\hat{k})$$

and

$$\vec{c} = \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$$

$$|\vec{a}| = \frac{1}{7}\sqrt{4+9+36}$$

$$= \frac{1}{7}\sqrt{49}$$

$$= \frac{7}{7} = 1 \qquad ½$$

$$|\vec{b}| = \frac{1}{7}\sqrt{36+9+4}$$

$$= \frac{1}{7}\sqrt{49}$$

$$= \frac{7}{7} = 1 \quad \frac{1}{2}$$

and $$|\vec{c}| = \frac{1}{7}\sqrt{9+36+4} = \frac{1}{7}\sqrt{49}$$

$$= \frac{7}{7} = 1 \quad \frac{1}{2}$$

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1 \quad \frac{1}{2}$$

Hence, vectors are unit vectors.

Q. 4. If $\vec{a} = 3\hat{i} + 7\hat{j} + 2\hat{k}$, $\vec{b} = 7\hat{i} - 2\hat{j} + 3\hat{k}$. Is $|\vec{a}| = |\vec{b}|$. Can we say $\vec{a} = \vec{b}$? Give reason. R&U

Sol. $$\vec{a} = 3\hat{i} + 7\hat{j} + 2\hat{k}$$

$$|\vec{a}| = \sqrt{9+49+4} = \sqrt{62} \quad 1$$

and $$\vec{b} = 7\hat{i} - 2\hat{j} + 3\hat{k}$$

$$|\vec{b}| = \sqrt{49+4+9} = \sqrt{62}$$

Thus, $$|\vec{a}| = |\vec{b}| \quad \frac{1}{2}$$

but $\vec{a} \neq \vec{b}$ because their corresponding components are different. ½

Q. 5. Find the vector of magnitude of 9 units in the direction of $\vec{a} - \vec{b}$ if $\vec{a} = 3\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} - \hat{k}$. R&U

Sol. Let, $$\vec{c} = \vec{a} - \vec{b}$$

$$= (3\hat{i} - 2\hat{j} + 3\hat{k}) - (\hat{i} - 4\hat{j} - \hat{k})$$

or $$\vec{c} = 2\hat{i} + 2\hat{j} + 4\hat{k}$$

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{2\hat{i} + 2\hat{j} + 4\hat{k}}{\sqrt{4+4+16}} \quad 1$$

$$= \frac{2\hat{i} + 2\hat{j} + 4\hat{k}}{\sqrt{24}}$$

$$= \frac{2(\hat{i} + \hat{j} + 2\hat{k})}{2\sqrt{6}}$$

$$\hat{c} = \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}$$

$$\therefore \text{ Required vector } = 9\hat{c} = \frac{9\hat{i} + 9\hat{j} + 18\hat{k}}{\sqrt{6}} \quad 1$$

Q. 6. If $|\vec{a} + \vec{b}| = 60$, $|\vec{a} - \vec{b}| = 40$ and $|\vec{a}| = 22$, then find $|\vec{b}|$. R&U **[S.Q.P. 2016-17]**

Sol. $$|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right) \quad 1$$

or $$|\vec{b}|^2 = 2600 - 484 = 2,116 \quad \frac{1}{2}$$

or $$|\vec{b}| = 46 \quad \frac{1}{2}$$

[CBSE Marking Scheme 2016]

Q. 7. The position vectors of points *A*, *B* and *C* are $\lambda\hat{i} + 3\hat{j}$, $12\hat{i} + \mu\hat{j}$ and $11\hat{i} - 3\hat{j}$ respectively. If *C* divides the line segment joining *A* and *B* in the ratio 3 : 1, find the values of λ and μ. R&U **[Delhi Comptt. 2017]**

Sol. $$11\hat{i} - 3\hat{j} = \frac{3(12\hat{i} + \mu\hat{j}) + 1(\lambda\hat{i} + 3\hat{j})}{4} \quad 1$$

$$44 = 36 + \lambda, -12 = 3\mu + 3$$

$$\lambda = 8, \mu = -5 \quad \frac{1}{2} + \frac{1}{2}$$

[CBSE Marking Scheme, 2017]

Long Answer Type Questions-I

(4 marks each)

Q. 1. Find a vector of magnitude 5 units and parallel to the resultant of $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$. R&U **[Delhi 2011]**

Sol. Given, $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Now, resultant of above vectors = $\vec{a} + \vec{b}$

$$= (2\hat{i} + 3\hat{j} - \hat{k}) + (\hat{i} - 2\hat{j} + \hat{k}) = 3\hat{i} + \hat{j} \quad 1$$

Let $$\vec{a} + \vec{b} = \vec{c}$$

$$\therefore \quad \vec{c} = 3\hat{i} + \hat{j}$$

Now unit vector $\hat{c}$ in the direction of $\vec{c}$ is $\frac{\vec{c}}{|\vec{c}|}$

$$= \frac{3\hat{i} + \hat{j}}{\sqrt{(3)^2 + (1)^2}} \quad 1$$

$$= \frac{3\hat{i} + \hat{j}}{\sqrt{10}} = \frac{3}{\sqrt{10}}\hat{i} + \frac{1}{\sqrt{10}}\hat{j} \quad 1$$

Hence, vector of magnitude 5 units and parallel to resultant of $\vec{a}$ and $\vec{b}$ is.

$$5\left(\frac{3}{\sqrt{10}}\hat{i}+\frac{1}{\sqrt{10}}\hat{j}\right) \text{ or } \frac{15}{\sqrt{10}}\hat{i}+\frac{5}{\sqrt{10}}\hat{j}. \qquad 1$$

Q. 2. Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$, and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$ Find a vector of magnitude 6 units, which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

R&U [All India 2010]

Sol. Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 4\hat{i}-2\hat{j}+3\hat{k}$

and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$

$\therefore\ 2\vec{a}-\vec{b}+3\vec{c}$

$= 2(\hat{i}+\hat{j}+\hat{k})-(4\hat{i}-2\hat{j}+3\hat{k})+3(\hat{i}-2\hat{j}+\hat{k})$

$= 2\hat{i}+2\hat{j}+2\hat{k}-4\hat{i}+2\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k}$

or $2\vec{a}-\vec{b}+3\vec{c} = \hat{i}-2\hat{j}+2\hat{k}$ **1½**

Now, a unit vector in the direction of vector

$$2\vec{a}-\vec{b}+3\vec{c} = \frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{(1)^2+(-2)^2+(2)^2}}$$

$$= \frac{\hat{i}-2\hat{j}+2\hat{k}}{\sqrt{9}}$$

$$= \frac{\hat{i}-2\hat{j}+2\hat{k}}{3} \qquad 1½$$

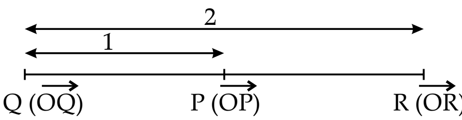
Hence, vector of magnitude 6 units parallel to the vector $2\vec{a}-\vec{b}+3\vec{c} = 6\left(\frac{1}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$

$$= 2\hat{i}-4\hat{j}+4\hat{k} \qquad 1$$

Q. 3. Find the position vector of a point *R*, which divides the line joining two points *P* and *Q* whose position vectors are $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively, externally in the ratio 1 : 2 Also, show that *P* is the mid-point, of line segment *RQ*.

R&U [HOTS; Delhi 2010]

Sol. Given, $\vec{OP}$ = Position vector of $P = 2\vec{a}+\vec{b}$ and $\vec{OQ}$ = Position vector of $Q = \vec{a}-3\vec{b}$ Let $\vec{OR}$ be the position vector of point *R*, which divides *PQ* in the ratio 1 : 2 externally.



$$\therefore \quad \vec{OR} = \frac{1(\vec{a}-3\vec{b})-2(2\vec{a}+\vec{b})}{1-2} \qquad 1$$

$$\left[\because \vec{OR} = \frac{m(\vec{OQ})-n(\vec{OP})}{m-n}. \text{Here}, m=1, n=2\right]$$

$$= \frac{\vec{a}-3\vec{b}-4\vec{a}-2\vec{b}}{-1}$$

$$= \frac{-3\vec{a}-5\vec{b}}{-1} = 3\vec{a}+5\vec{b}$$

Hence, $\vec{OR} = 3\vec{a}+5\vec{b}$ **1½**

Now, we have to show that *P* is the mid-point of *RQ*,

i.e. $\vec{OP} = \dfrac{\vec{OR}+\vec{OQ}}{2}$

We have, $\vec{OR} = 3\vec{a}+5\vec{b}$, $\vec{OQ} = \vec{a}-3\vec{b}$

$$\therefore \quad \frac{\vec{OR}+\vec{OQ}}{2} = \frac{(3\vec{a}+5\vec{b})+(\vec{a}-3\vec{b})}{2}$$

$$= \frac{4\vec{a}+2\vec{b}}{2} = \frac{2(2\vec{a}+\vec{b})}{2}$$

$$= 2\vec{a}+\vec{b} = \vec{OP} \quad [\because \vec{OP} = 2\vec{a}+\vec{b}]$$

Hence, *P* is the mid-point of line segment *RQ*. **1½**

Q. 4. Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k})$, $B(\hat{i}+2\hat{j}+3\hat{k})$ and $C(7\hat{i}-\hat{k})$ are collinear. **R&U [Delhi 2009C]**

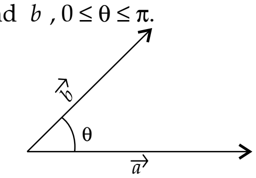
Sol. Try yourself

TOPIC-2
Dot Product of Vectors

Revision Notes

1. Products of Two Vectors and Projection of Vectors

(a) Scalar Product or Dot Product : The dot product of two vectors $\vec{a}$ and $\vec{b}$ is defined by, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.



Consider $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}.\vec{b} = a_1b_1+a_2b_2+a_3b_3$.

➲ **Projection of a vector :** $\vec{a}$ on the other vector say $\vec{b}$ is given as $\left(\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\right)$.

➲ **Projection of a vector :** $\vec{b}$ on the other vector say $\vec{a}$ is given as $\left(\dfrac{\vec{a}.\vec{b}}{|\vec{a}|}\right)$.

Know the Properties (Dot Product)

- **Properties/Observations of Dot product**

➲ $\hat{i}.\hat{i} = |\hat{i}||\hat{i}|\cos 0 = 1$ or $\hat{i}.\hat{i} = 1 = \hat{j}.\hat{j} = \hat{k}\cdot\hat{k}$

➲ $\hat{i}.\hat{j} = |\hat{i}||\hat{j}|\cos\frac{\pi}{2} = 0$ or $\hat{i}.\hat{j} = 0 = \hat{j}.\hat{k} = \hat{k}\cdot\hat{i}$

➲ $\vec{a}\cdot\vec{b} \in R$, where R is real number *i.e.*, any scalar.

➲ $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ (Commutative property of dot product).

➲ $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ or $|\vec{a}| = 0$ or $|\vec{b}| = 0$.

➲ If $\theta = 0$, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$. Also $\vec{a}.\vec{a} = |\vec{a}|^2 = a^2$; as θ in this case is 0.

Moreover if $\theta = \pi$, then $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$.

➲ $\vec{a}.\left(\vec{b}+\vec{c}\right) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$ (Distributive property of dot product).

➲ $\vec{a}.\left(-\vec{b}\right) = -\left(\vec{a}.\vec{b}\right) = \left(-\vec{a}\right).\vec{b}$.

Know the Formulae

➲ **Angle between two vectors** $\vec{a}$ and $\vec{b}$ can be found by the expression given below :

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \text{ or, } \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right).$$

Objective Type Questions

(1 mark each)

Q. 1. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when

(a) $0 < \theta < \frac{\pi}{2}$ (b) $0 \leq \theta \leq \frac{\pi}{2}$

(c) $0 < \theta < \pi$ (d) $0 \leq \theta \leq \pi$

[NCERT Misc. Ex. Q. 16, Page 459]

Ans. Correct option : (b)

Explanation :

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$. Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

It is known that, $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$\therefore \quad \vec{a}.\vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

$\Rightarrow \cos\theta \geq 0 \quad \left[\because |\vec{a}| \text{ and } |\vec{b}| \text{ are positive.}\right]$

$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$.

Q. 2. Let $\vec{a}$ and $\vec{b}$ be two-unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if

(a) $\theta = \frac{\pi}{4}$ (b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{2}$ (d) $\theta = \frac{2\pi}{2}$

[NCERT Misc. Ex. Q. 17, Page 459]

Ans. Correct option : (d)

Explanation :

Let $\vec{a}$ and $\vec{b}$ be two-unit vectors and θ be the angle between them.

Then, $|\vec{a}+\vec{b}| = |\vec{b}| = 1$.

Now, $\vec{a}+\vec{b}$ is a unit vector if $|\vec{a}+\vec{b}| = 1$.

$$|\vec{a}+\vec{b}| = 1$$

$\Rightarrow (\vec{a}+\vec{b})^2 = 1$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}\vec{b} = 1$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$

$\Rightarrow 1^2 + 2|\vec{a}||\vec{b}|\cos\theta + 1^2 = 1$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \frac{2\pi}{3}$

So that, $|\vec{a}+\vec{b}|$ is a unit vector if $\theta = \frac{2\pi}{3}$.

Q. 3. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4, respectively, and $\vec{a}.\vec{b} = 2\sqrt{3}$ is :

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{5\pi}{2}$

[NCERT Exemp. Ex. 10.3, Q. 22, Page 217]

Ans. Correct option : (b)

Explanation :

Here, $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 4$ and $\vec{a}.\vec{b} = 2\sqrt{3}$ [Given]

We know that,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow 2\sqrt{3} = \sqrt{3}.4.\cos\theta$

$\Rightarrow \cos\theta = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2}$

$\therefore \quad \theta = \frac{\pi}{3}$

Q. 4. Find the value of λ such that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal.

(a) 0 (b) 1

(c) $\frac{3}{2}$ (d) $\frac{-5}{2}$

[NCERT Exemp. Ex. 10.3, Q. 23, Page 217]

Ans. Correct option : (d)

Explanation :

Since, two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal,

i.e., $\vec{a}.\vec{b} = 0$

$\therefore (2\hat{i} + \lambda\hat{j} + \hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) = 0$

$\Rightarrow \quad 2 + 2\lambda + 3 = 0$

$\therefore \quad \lambda = \frac{-5}{2}$

Q. 5. The value of λ for which the vectors $3\hat{i} - 6\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + \lambda\hat{k}$ are parallel is

(a) 2/3 (b) 3/2

(c) 5/2 (d) 2/5

[NCERT Exemp. Ex. 10.3, Q. 24, Page 217]

Ans. Correct option : (a)

Explanation :

Let $\vec{a} = 3\hat{i} - 6\hat{j} + \hat{k}$ and $\vec{b} = 2\vec{i} - 4\vec{j} + \lambda\hat{k}$

Since, $\vec{a} \,||\, \vec{b}$

$\Rightarrow \frac{3}{2} = \frac{-6}{-4} = \frac{1}{\lambda}$

$\Rightarrow \lambda = \frac{2}{3}$

Q. 6. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ is

(a) 1 **(b)** 3
(c) –3/2 **(d)** None of these

[NCERT Exemp. Ex. 10.3, Q. 29, Page 218]

Ans. Correct option : (c)

Explanation :

We have, $\vec{a}+\vec{b}+\vec{c}=0$ and $\vec{a}^2=1$, $\vec{b}^2=1$, $\vec{c}^2=1$

$\therefore\ (\vec{a}+\vec{b}+\vec{c})(\vec{a}+\vec{b}+\vec{c})=0$

$\Rightarrow\ \vec{a}^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}^2+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}^2=0$

$\Rightarrow\ \vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$[\because\ \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a},\ \vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{b}\ and\ \vec{c}\cdot\vec{a}=\vec{a}\cdot\vec{c}]$

$\Rightarrow\ 1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\dfrac{3}{2}$

Q. 7. The projection vector of $\vec{a}$ on $\vec{b}$ is

(a) $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\vec{b}$ **(b)** $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

(c) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$ **(d)** $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\hat{b}$

[NCERT Exemp. Ex. 10.3, Q. 30, Page 218]

Ans. Correct option : (a)

Explanation :

Projection vector of $\vec{a}$ on $\vec{b}$ is given by,

$$=\vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}\vec{b}$$

$$=\left(\vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}\right)\cdot\vec{b}$$

Q. 8. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=2$, $|\vec{b}|=3$ and $|\vec{c}|=5$, then the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ is

(a) 0 **(b)** 1
(c) –19 **(d)** 38

[NCERT Exemp. Ex. 10.3, Q. 31, Page 218]

Ans. Correct option : (c)

Explanation :

Here, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $\vec{a}^2=4$, $\vec{b}^2=9$, $\vec{c}^2=25$

$\therefore\ (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}$

$\Rightarrow\ \vec{a}^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}^2+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}^2=\vec{0}$

$\Rightarrow\ \vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0\ [\because\ \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$

$\Rightarrow\ 4+9+25+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow\ \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\dfrac{-38}{2}=-19$

Very Short Answer Type Questions

(1 mark each)

Q. 1. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$, so that $\sqrt{2}\,\vec{a}-\vec{b}$ is a unit vector ? A **[Delhi Set I, II, III Comptt. 2015]**

Sol.

$$2\,\vec{a}.\vec{a}-2\sqrt{2}\,\vec{a}.\vec{b}+\vec{b}.\vec{b}\ =1$$

$$2\,|\vec{a}|^2-2\sqrt{2}\,\vec{a}.\vec{b}+|\vec{b}|^2\ =1$$

$$2-2\sqrt{2}\,\vec{a}.\vec{b}+1\ =1$$

$$\vec{a}.\vec{b}\ =\frac{-2}{-2\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$|\vec{a}|\,|\vec{b}|\cos\theta=\frac{1}{\sqrt{2}}$$

$$1.1.\cos\theta=\frac{1}{\sqrt{2}}$$

$$\theta=\frac{\pi}{4}$$ 1

[CBSE Marking Scheme 2015]

Q. 2. Find the projection of vector $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ on the vector $\vec{b}=2\hat{i}+2\hat{j}+\hat{k}$. R&U **[NCERT]**

[O.D. Set I, II, III Comptt. 2015]

Sol. $(2\hat{i}+3\hat{j}+2\hat{k})\cdot(2\hat{i}+2\hat{j}+\hat{k})\ =12$ ½

$$p=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\quad\text{or}\quad p=\frac{12}{|\vec{b}|}$$

$$=\frac{12}{3}=4$$ ½

[CBSE Marking Scheme 2015]

Q. 3. Write the projection of the vector $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ on the vector $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$.

R&U **[Delhi Set I, II, III Comptt. 2014]**

Sol. Projection of a vector $\vec{a}$ on the vector $\vec{b}$ is given by

$$=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$=\frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{1^2+2^2+2^2}}$$

$$=\frac{2-2+2}{3}=\frac{2}{3}$$ 1

[CBSE Marking Scheme 2014]

Commonly Made Error

- Most of the candidates calculate dot product instead of applying projection formula.

Answering Tip

- Clarify the concept of scalar projection of vector thoroughly.

Q. 4. Find the projection of vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$. R&U **[Delhi Set II, III 2014]**

Sol. Required projection :

$$= \frac{(\hat{i}+3\hat{j}+7\hat{k}).(2\hat{i}-3\hat{j}+6\hat{k})}{|2\hat{i}-3\hat{j}+6\hat{k}|}$$

$$= \frac{35}{\sqrt{49}} = \frac{35}{7} = 5 \qquad 1$$

Q. 5. Write the projection of the vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$. R&U **[Foreign Set I, II, III 2014]**

Sol. We know that the projection of a vector $\vec{a}$ on the vector $\vec{b}$ is given by $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

$\therefore$ The projection of the vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$ is,

$$\left(\hat{i}+\hat{j}+\hat{k}\right).\left(\frac{\hat{j}}{\sqrt{0^2+1^2+0^2}}\right) = 1. \qquad 1$$

Q. 6. Write the projection of the vector $7\hat{i}+\hat{j}-4\hat{k}$ on the vector $2\hat{i}+6\hat{j}+3\hat{k}$. R&U **[Delhi 2015]**

[Delhi Set I, II, III Comptt. 2013]

Sol. Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(7\hat{i}+\hat{j}-4\hat{k})\cdot(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{2^2+6^2+3^2}}$$

$$= \frac{14+6-12}{\sqrt{49}} = \frac{8}{7} \qquad 1$$

[CBSE Marking Scheme 2013]

Q.7. Write the projection of $\vec{b}+\vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$.

R&U **[O.D. Set I, II, III Comptt. 2013]**

Sol.

$$\vec{a} = 2\hat{i}-2\hat{j}+\hat{k}$$

$$\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$$

and $$\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$$

$$\vec{b}+\vec{c} = \hat{i}+2\hat{j}-2\hat{k}+2\hat{i}-\hat{j}+4\hat{k}$$

$$= 3\hat{i}+\hat{j}+2\hat{k}$$

Projection of $(\vec{b}+\vec{c})$ on $\vec{a}$

$$= \frac{(\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}|} \qquad ½$$

$$= \frac{(3\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-2\hat{j}+\hat{k})}{\sqrt{4+4+1}}$$

$$= \frac{6-2+2}{3} = 2. \qquad ½$$

[CBSE Marking Scheme 2013]

Q. 8. Find 'λ' when the projection of $\vec{a} = \lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units.

R&U **[Delhi Set I, II, III 2012]**

Sol. Projection of $\vec{a}$ on $\vec{b}$ = 4, (given)

$\therefore$ $$\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 4 \qquad ½$$

or $$\frac{(\lambda\hat{i}+\hat{j}+4\hat{k}).(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{4+36+9}} = 4$$

or $$2\lambda + 6 + 12 = 7 \times 4$$

or $$2\lambda = 28 - 18 = 10$$

or $$\lambda = \frac{10}{2} = 5. \qquad ½$$

[CBSE Marking Scheme 2012]

Q. 9. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}|$ = 13 and $|\vec{a}|$ = 5, find the value of $|\vec{b}|$.

A **[O.D. Set III 2014]**

Sol. $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}$

or $(13)^2 = (5)^2+|\vec{b}|^2+0$

$\{\because \vec{a}\perp\vec{b} \Rightarrow |\vec{a}|\cdot|\vec{b}|\cos\theta = 0 \text{ as } \theta = 90°\}$

or $(169-25) = |\vec{b}|^2$

or $|\vec{b}| = 12$ **1**

[CBSE Marking Scheme 2014]

Q. 10. Write the value of λ so that the vectors $\vec{a} = 2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other ?

A **[Delhi Set I, II, III Comptt. 2013]**
[O.D. Set I, II, III Comptt. 2012]

Sol. $\vec{a} = 2\hat{i}+\lambda\hat{j}+\hat{k},$

and $\vec{b} = \hat{i}-2\hat{j}+3\hat{k}$

For perpendicular :

$\vec{a}.\vec{b} = 0$

or $2\times 1+\lambda(-2)+1\times 3 = 0$ ½

or $2\lambda = 5$

or $\lambda = \frac{5}{2}\cdot$ ½

Q. 11. For what value of λ are the vectors $\hat{i}+2\lambda\hat{j}+\hat{k}$ and $2\hat{i}+\hat{j}-3\hat{k}$ perpendicular ?

R&U **[O.D. Set I, II, III Comptt. 2013, 2011]**
[Delhi Set I, II, III Comptt. 2012]

Sol. For two vectors to be perpendicular, their product should be zero.

$\therefore (\hat{i}+2\lambda\hat{j}+\hat{k}).(2\hat{i}+\hat{j}-3\hat{k}) = 0$

or $1\times 2+2\lambda\times 1+1\times(-3) = 0$

$\lambda = \frac{1}{2}\cdot$ **1**

Q. 12. Find $|\vec{x}|$, if for a unit vector $\vec{a}$ and $(\vec{x}+\vec{a}).(\vec{x}-\vec{a}) = 15.$ A **[O.D. Set I 2013]**

Sol. $|\vec{a}| = 1$

Given $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$

or $\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}+\vec{x}\cdot\vec{x}-\vec{a}\cdot\vec{a} = 15$

or $|\vec{x}|^2-1 = 15$

$\{\because a \text{ is a unit vector } |\vec{a}| = 1\}$

$|\vec{x}|^2 = 16$ or $|\vec{x}|\ \pm 4$

As magnitude of a vector is non-negative.

So $|\vec{x}| = 4.$ **1**

[CBSE Marking Scheme 2013]

Q. 13. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\vec{a}-\sqrt{2}\vec{b}$ to be unit vectors ?

R&U **[O.D. Set-II, 2016]**

Sol. $\bar{a}-\sqrt{2}\bar{b}$ is a unit vector

$|\bar{a}-\sqrt{2}\bar{b}| = 1$

$\therefore |\bar{a}-\sqrt{2}\bar{b}|^2 = 1$

$|\bar{a}|^2+|\sqrt{2}\bar{b}|^2-2\sqrt{2}\bar{a}\cdot\bar{b} = 1$ $(|\bar{a}|=|\bar{b}|=1)$ (θ: Angle between vectors $\bar{a}$ & $\bar{b}$)

$1+2-2\sqrt{2}\bar{a}\cdot\bar{b} = 1$ $\bar{a}\cdot\bar{b} = \frac{1}{\sqrt{2}} \Rightarrow |\bar{a}||\bar{b}|\cos\theta = \frac{1}{\sqrt{2}}$ $\therefore \theta = 45°$ (Angle between $\bar{a}$ & $\bar{b}$ is 45°)

[Topper's Answer 2016]

Q. 14. If $\hat{a},\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find value of $|2\hat{a}+\hat{b}+\hat{c}|$.

R&U **[All India 2015]**

Sol. Given $\hat{a},\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, *i.e.*,

$\hat{a}.\hat{b} = \hat{b}.\hat{c} = \hat{c}.\hat{a} = 0$ ...(i)

and $|\hat{a}| = |\hat{b}| = |\hat{c}| = 1$ ...(ii)

Now, $|2\hat{a}+\hat{b}+\hat{c}|^2 = (2\hat{a}+\hat{b}+\hat{c}).(2\hat{a}+\hat{b}+\hat{c})$

$= 4(\hat{a}.\hat{a})+2(\hat{a}.\hat{b})+2(\hat{a}.\hat{c})+2(\hat{b}.\hat{a})$

$+(\hat{b}.\hat{b})+(\hat{b}.\hat{c})+2(\hat{c}.\hat{a})+(\hat{c}.\hat{b})+(\hat{c}.\hat{c})$

$[\because \vec{a}.\vec{a} = |\vec{a}|^2]$

[$\because$ dot product is distributive over addition] ½

$$= 4(|\hat{a}|^2)+2(0)+2(0)+2(0)+|\hat{b}|^2+(0)$$
$$+2(0)+(0)+|\hat{c}|^2$$

[from Eq. (i) and $\vec{a}.\vec{b} = \vec{b}.\vec{a}$]

$$= 4(1) + 1 + 1 = 4 + 1 + 1 = 6$$

$\therefore \quad |2\hat{a}+\hat{b}+\hat{c}| = \sqrt{6}$ ½

[$\because$ length cannot be negative]

[CBSE Marking Scheme 2015]

Q. 15. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. R&U **[Delhi 2014]**

Sol. Given, $|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$

Now, $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$= \vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{b}$

or $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2$

[$\because \vec{a}.\vec{b} = \vec{b}.\vec{a}$ and $\vec{a}.\vec{a} = |\vec{a}|^2$]

or $1 = 1 + 2\vec{a}.\vec{b}+1$

or $2\vec{a}.\vec{b} = -1$ ½

or $|\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2}$ [$\because \vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$]

or $\cos\theta = -\frac{1}{2}$ [$\because |\vec{a}| = |\vec{b}| = 1$]

or $\cos\theta = \cos\frac{2\pi}{3}$ or $\theta = \frac{2\pi}{3}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$. ½

[CBSE Marking Scheme 2014]

Q. 16. Find $\vec{a}.\vec{b}$, if $\vec{a} = -\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}-\hat{k}$. R&U **[All India 2009C]**

Sol. Given, $\vec{a} = -\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}-\hat{k}$

Then, $\vec{a}.\vec{b} = (-\hat{i}+\hat{j}-2\hat{k}).(2\hat{i}+3\hat{j}-\hat{k})$

$= -2 + 3 + 2 = 3$ 1

[CBSE Marking Scheme 2019]

Q. 17. If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a}.\vec{b} = 3$, then find the projection of $\vec{b}$ on $\vec{a}$. R&U **[All India 2010C]**

Sol. Given, $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a}.\vec{b} = 3$

$\therefore$ Projection of $\vec{b}$ on $\vec{a}$

$$= \frac{\vec{b}.\vec{a}}{|\vec{a}|} = \frac{\vec{a}.\vec{b}}{|\vec{a}|} \quad [\because \vec{a}.\vec{b} = \vec{b}.\vec{a}]$$

$$= \frac{3}{2} \quad [\because \vec{a}.\vec{b} = 3 \text{ and } |\vec{a}| = 2]$$ 1

Q. 18. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ then write the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$. R&U **[Foreign 2015]**

Sol. Try Yourself 1

Q. 19. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\vec{a}-\vec{b})$ is a unit vector. R&U **[Delhi 2014C][NCERT Exemplar]**

Sol. Try Yourself 1

Q. 20. Write the projection of vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$. R&U **[All India 2011]**

Sol. Try Yourself 1

Q. 21. If $\hat{P}$ is a unit vector and $(\vec{x}-\hat{P}).(\vec{x}+\hat{P}) = 80$, then find $|\vec{x}|$. R&U **[All India 2009]**

Sol. Try Yourself 1

Q. 22. Write the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, having $\vec{a}.\vec{b} = \sqrt{6}$. R&U **[All India 2011]**

Sol. Try Yourself 1

Q. 23. If $|\vec{a}| = \sqrt{3}, |\vec{b}| = 2$ and angle between $\vec{a}$ and $\vec{b}$ is 60°, then find $\vec{a}.\vec{b}$. R&U **[Delhi 2011C]**

Sol. Try Yourself 1

Q. 24. Find $\vec{a}.\vec{b}$, if $\vec{a} = 3\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = 2\hat{i}+3\hat{j}+3\hat{k}$. R&U **[Delhi 2009C]**

Sol. Try Yourself 1

Q. 25. Find the value of λ, if the vectors $3\hat{i}+\lambda\hat{j}+3\hat{k}$ and $3\hat{i}+2\hat{j}-4\hat{k}$ are perpendicular to each other. R&U **[All India 2010C]**

Sol. Try Yourself 1

Q. 26. Find the projection of $\vec{a}$ on $\vec{b}$, if $\vec{a}.\vec{b} = 8$ and $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$. R&U **[Delhi 2009]**

Sol. Try Yourself 1

Q. 27. Find the magnitude of each of the vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. R&U **[Delhi/O.D. 2018]**

Sol. $|\vec{a}| = |\vec{b}| = 3$ ½+½

[CBSE Marking Scheme, 2018]

Detailed Solution :

Given, angle $\theta = 60°$ and $\vec{a}.\vec{b} = \frac{9}{2}$

$$\because \quad \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$$

$$\Rightarrow \quad \cos 60° = \frac{9/2}{|\vec{a}|.|\vec{a}|} \quad (\because |\vec{a}| = |\vec{b}|) \quad ½$$

$$\Rightarrow \quad \frac{1}{2}|\vec{a}|^2 = \frac{9}{2} \Rightarrow |\vec{a}| = 3.$$

$$\therefore \quad |\vec{a}| = |\vec{b}| = 3. \quad ½$$

Short Answer Type Questions **(2 marks each)**

Q. 1. If either vector $\vec{a} = 0$ or $\vec{b} = 0$ then $\vec{a}\cdot\vec{b} = 0$ But the converse need not be true. Justify your answer. A **[NCERT] [Foreign 2011]**

Sol. Let $\vec{a}\cdot\vec{b} = 0$

or $|\vec{a}||\vec{b}|\cos\theta = 0$

as $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ 1

Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$

or $\cos\theta = 0$

$\theta = \dfrac{\pi}{2}$ 1

thus, it is clear that the dot product of two non-zero perpendicular vector is always zero. This shows that the converse is not true.

Q. 2. If $\vec{a}, \vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{a}|$, then prove that $2\vec{a}+\vec{b}$ is perpendicular to $\vec{b}$. A **[Delhi Set I 2013]**

Sol. Given, $|\vec{a}+\vec{b}| = |\vec{a}|$

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2$

or $|\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = |\vec{a}|^2$

$|\vec{b}|^2 + 2\vec{a}\cdot\vec{b} = 0$ ...(i)

Now, $(2\vec{a}+\vec{b})\cdot\vec{b} = 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b}$ 1

$\vec{b}.(2\vec{a}+\vec{b}) = 0$ 1

From eqn. (i), $2\vec{a}+\vec{b}$ is $\perp$ to $\vec{b}$.

[CBSE Marking Scheme, 2013]

Q. 3. Find the projection (vector) of $2\hat{i}-\hat{j}+\hat{k}$ on $\hat{i}-2\hat{j}+\hat{k}$. R&U **[SQP 2017-18]**

Sol. $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}, \vec{b} = \hat{i}-2\hat{j}+\hat{k}, \vec{a}.\vec{b} = 5, |\vec{b}| = \sqrt{6}.$ ½

The required projection (vector) of $\vec{a}$ on $\vec{b}$

$= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|^2}\vec{b}$ 1

$= \dfrac{5}{6}(\hat{i}-2\hat{j}+\hat{k}).$ ½

[CBSE Marking Scheme, 2017]

Q. 4. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find sin θ. R&U **[Delhi/O.D.-2018]**

Sol. $\sin\theta = \dfrac{|(\hat{i}-2\hat{j}+3\hat{k})\times(3\hat{i}-2\hat{j}+\hat{k})|}{|\hat{i}-2\hat{j}+3\hat{j}||3\hat{i}-2\hat{j}+\hat{k}|}$ ½

$|(\hat{i}-2\hat{j}+3\hat{k})\times(3\hat{i}-2\hat{j}+\hat{k})|$

$= |4\hat{i}+8\hat{j}+4\hat{k}| = 4\sqrt{6}$ 1

$\sin\theta = \dfrac{4\sqrt{6}}{14} = \dfrac{2\sqrt{6}}{7}$ ½

[CBSE Marking Scheme, 2018]

Detailed Answer :

Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k} \Rightarrow |\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

$\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k} \Rightarrow |\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\because \quad \cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$

$= \dfrac{(\hat{i} - 2\hat{j} + 3\hat{k}).(3\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{14}.\sqrt{14}}$ **1**

$= \dfrac{3+4+3}{14} = \dfrac{10}{14} = \dfrac{5}{7}.$

$\therefore \quad \sin\theta = \sqrt{1 - \cos^2\theta}$

$\sin\theta = \sqrt{1 - \dfrac{25}{49}}$

$= \dfrac{\sqrt{24}}{7} = \dfrac{2\sqrt{6}}{7}$ **1**

Long Answer Type Questions-I

(4 marks each)

Q. 1. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha} = 4\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{\beta} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{p}.\vec{q} = 21$, where $\vec{q} = 3\hat{i} + \hat{j} - \hat{k}$.

A [O.D. Set I, II, III Comptt. 2014]

Sol. Any vector perpendicular to both $\vec{\alpha}$ and $\vec{\beta}$ = Parallel to $(\vec{\alpha} \times \vec{\beta})$ ½

$\therefore \quad \vec{p} = \lambda(\vec{\alpha} \times \vec{\beta})$

$= \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 5 & -1 \\ 1 & -4 & 5 \end{vmatrix}$ ½

$= \lambda[\hat{i}(25-4) - \hat{j}(20+1) + \hat{k}(-16-5)]$

$= \lambda[21\hat{i} - 21\hat{j} - 21\hat{k}]$ **1**

$\vec{p}.\vec{q} = 21$ (Given)

$\lambda(21\hat{i} - 21\hat{j} - 21\hat{k}).(3\hat{i} + \hat{j} - \hat{k}) = 21$

$\lambda(63 - 21 + 21) = 21$ ½

$\lambda = \dfrac{1}{3}$ ½

$\therefore \quad p = \lambda(21\hat{i} - 21\hat{j} - 21\hat{k})$

$p = \dfrac{1}{3}(21\hat{i} - 21\hat{j} - 21\hat{k})$

$p = 7\hat{i} - 7\hat{j} - 7\hat{k}$ **1**

[CBSE Marking Scheme 2014]

Q. 2. Find the unit vector perpendicular to the plane *ABC* where the position vectors *A*, *B* and *C* are $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + \hat{j} + 2\hat{k}$ and $2\hat{i} + 3\hat{k}$.

R&U [O.D. Set I, II, III Comptt. 2014]

Sol. Required vector = $\dfrac{\vec{AB} \times \vec{AC}}{\left|\vec{AB} \times \vec{AC}\right|}$ **1**

or $\vec{AB}$ = (Position vector of *B*) – (Position vector of *A*)

or $\vec{AB} = -\hat{i} + 2\hat{j} + \hat{k}$

Similarly

$\vec{AC} = 0\hat{i} + \hat{j} + 2\hat{k}$ **1**

$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

$= \hat{i}(4-1) - \hat{j}(-2-0) + \hat{k}(-1-0)$

$\vec{AB} \times \vec{AC} = 3\hat{i} + 2\hat{j} - \hat{k}$ **1**

$\therefore$ Required unit vector

$= \dfrac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} - \hat{k})$ **1**

[CBSE Marking Scheme 2014]

Commonly Made Error

- Sometimes, candidates use cross product without evaluating $\vec{AB}$ and $\vec{AC}$. Some candidates make mistakes while evaluating the unit vector in the final answer.

Answering Tip

- Vector algebra in finding unit vector need to be understood by the students.

Q. 3. If $\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$, then find the value of λ so that $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular vectors. **R&U [O.D. Set I 2013]**

Sol. Given,

$$\vec{a} = \hat{i} - \hat{j} + 7\hat{k}$$

and $$\vec{b} = 5\hat{i} - \hat{j} + \lambda\hat{k}$$

$$\vec{a} + \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) + (5\hat{i} - \hat{j} + \lambda\hat{k}) \quad 1$$

$$= 6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}$$

and $$\vec{a} - \vec{b} = (\hat{i} - \hat{j} + 7\hat{k}) - (5\hat{i} - \hat{j} + \lambda\hat{k})$$

$$= -4\hat{i} + (7-\lambda)\hat{k} \quad 1$$

Since $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular vectors,

$$\therefore \quad (\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = 0$$

$$\{6\hat{i} - 2\hat{j} + (7+\lambda)\hat{k}\}\cdot\{-4\hat{i} + (7-\lambda)\hat{k}\} = 0$$

or $-24 + (7+\lambda)(7-\lambda) = 0$ 1

or $49 - \lambda^2 - 24 = 0$

or $\lambda^2 = 49 - 24 = 25$

or $\lambda = \pm 5$ units 1

[CBSE Marking Scheme, 2013]

Q. 4. Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$. R&U **[Delhi Set I, II, III, 2014]**

Sol. $\vec{a} + \vec{b} + \vec{c} = 0, \therefore \vec{a} + \vec{b} = -\vec{c}$ ½

or $\left(\vec{a} + \vec{b}\right)^2 = \left(-\vec{c}\right)^2 = |\vec{c}|^2$ ½

or $9 + 25 + 2|\vec{a}||\vec{b}|\cos\theta = 49$ 1

θ being angle between $\vec{a}$ and $\vec{b}$, 1

$\therefore \cos\theta = \dfrac{15}{2\cdot3\cdot5} = \dfrac{1}{2}$ or $\theta = \dfrac{\pi}{3}$ 1

[CBSE Marking Scheme 2014]

AI **Q. 5. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$.** R&U **[OD 2009]**

[O.D. Set I, II, III, 2014]

Sol. Given that

$$\vec{a}\cdot\frac{\vec{b}+\vec{c}}{\left|\vec{b}+\vec{c}\right|} = 1 \quad ½$$

or $$\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = \left|\vec{b}+\vec{c}\right| \quad ½$$

$$(\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}+4\hat{j}-5\hat{k}) + (\hat{i}+\hat{j}+\hat{k})\cdot(\lambda\hat{i}+2\hat{j}+3\hat{k})$$

$$= \left|(\lambda+2)\hat{i} + 6\hat{j} - 2\hat{k}\right| \quad ½$$

or $(2 + 4 - 5) + (\lambda + 2 + 3)$

$$= \sqrt{(\lambda+2)^2 + 36 + 4} \quad 1$$

$\therefore \quad (\lambda + 6)^2 = (\lambda + 2)^2 + 40$ or $\lambda = 1$ ½

Hence $$\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|} = \frac{3\hat{i}+6\hat{j}-2\hat{k}}{7}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k} \quad 1$$

[CBSE Marking Scheme 2014]

Alternative Method :

Let $$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

and $$\vec{b} + \vec{c} = (2\hat{i}+4\hat{j}-5\hat{k}) + (\lambda\hat{i}+2\hat{j}+3\hat{k})$$

$$\vec{b} + \vec{c} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

So, $$|\vec{b} + \vec{c}| = \sqrt{(2+\lambda)^2 + 36 + 4}$$

$$= \sqrt{\lambda^2 + 4\lambda + 44} = r, \text{ (say) ...(i)} \quad ½$$

$\therefore$ Unit vector along $\vec{b} + \vec{c}$ is given by :

$$\frac{(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|} = \frac{2+\lambda}{r}\hat{i} + \frac{6}{r}\hat{j} - \frac{2}{r}\hat{k} \quad ½$$

Since, $\vec{a}\cdot(\vec{b}+\vec{c}) = 1$

or $(\hat{i}+\hat{j}+\hat{k})\cdot\left(\dfrac{2+\lambda}{r}\hat{i} + \dfrac{6}{r}\hat{j} - \dfrac{2}{r}\hat{k}\right) = 1$ ½

or $1\left(\dfrac{2+\lambda}{r}\right) + 1\left(\dfrac{6}{r}\right) + 1\left(\dfrac{-2}{r}\right) = 1$ ½

or $\dfrac{\lambda+6}{r} = 1 \Rightarrow \lambda + 6 = r$

or $\lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44}$, from (i) ½

or $\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

or $8\lambda = 8$

$\therefore \quad \lambda = 1 \Rightarrow r = 7$ ½

Hence, $$\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|} = \frac{3\hat{i}+6\hat{j}-2\hat{j}}{7}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k} \quad 1$$

Q. 6. Dot product of a vector with vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively 4, 0 and 2. Find the vector.

A **[Delhi Set I, II, III Comptt. 2013]**

Sol. Let the required vector be

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad ½$$

Also, let $\vec{a} = \hat{i} - \hat{j} + \hat{k}$,

$$\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$$

and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$

Given, $\vec{r}\cdot\vec{a} = 4$ or $x - y + z = 4$ ...(i) ½

$\vec{r}\cdot\vec{b} = 0$ or $2x + y - 3z = 0$ ...(ii) ½

and $\vec{r}\cdot\vec{c} = 2$ or $x + y + z = 2$ ...(iii) ½

By solving eqns. (i), (ii), & (iii), we get

$$x = 2, y = -1, z = 1 \quad 1½$$

$\therefore$ The req. vector is $\vec{r} = 2\hat{i} - \hat{j} + \hat{k}$ ½

[CBSE Marking Scheme 2013]

Answering Tip

- Generally students commit errors in simplifying equation which leads to get the wrong result.

Q. 7. Find the values of λ for which the angle between the vectors $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$ is obtuse. A **[O.D. Set I, II, III Comptt. 2013]**

Sol. Here, $\vec{a} = 2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}$

and $\vec{b} = 7\hat{i} - 2\hat{j} + \lambda\hat{k}$

If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Or $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$ ½

for θ to be obtuse

$$\cos\theta < 0 \text{ or } \vec{a}\cdot\vec{b} < 0 \quad ½$$

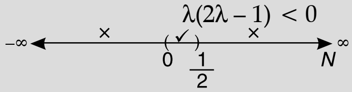
or $(2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k})\cdot(7\hat{i} - 2\hat{j} + \lambda\hat{k}) < 0$ 1

or $14\lambda^2 - 8\lambda + \lambda < 0$

or $14\lambda^2 - 7\lambda < 0$

or $2\lambda^2 - \lambda < 0$

or $\lambda(2\lambda - 1) < 0$ ½



½

$\therefore$ $\lambda \in \left(0, \frac{1}{2}\right)$ 1

[CBSE Marking Scheme 2013]

Q. 8. If the sum of two unit vectors $\vec{a}$ and $\vec{b}$ is a unit vector, then show that the magnitude of their difference is $\sqrt{3}$.

R&U **[Delhi Set I, II, III Comptt. 2012]**

Sol. Let, $\vec{c} = \vec{a} + \vec{b}$

or $|c| = \sqrt{|a|^2 + |b|^2 + 2|a||b|\cos\theta}$ 1

Given that $\vec{c}$, $\vec{a}$ and $\vec{b}$ are unit vectors.

So, $|a| = |b| = |c| = 1$

or $2|a||b|\cos\theta = -1$ 1

or $\vec{d} = \vec{a} - \vec{b}$

$|d| = \sqrt{|a|^2 + |b|^2 - 2|a||b|\cos\theta}$

$= \sqrt{1+1+1} = \sqrt{3}$ 1

$\therefore$ $|\vec{a} - \vec{b}| = \sqrt{3}$. 1

[CBSE Marking Scheme 2012]

Q. 9. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}| = 5, |\vec{b}| = 12$ and $|\vec{c}| = 13$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$.

R&U **[Delhi Set I, II, III, 2012]**

Sol. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

or $(\vec{a} + \vec{b} + \vec{c})^2 = 0$ ½

or $\vec{a}.\vec{a} + \vec{b}.\vec{b} + \vec{c}.\vec{c} + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

$= 0$ 1

or $|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$

$+ 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$ 1

$\therefore$ $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

$= -\frac{1}{2}[|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2]$ ½

$= -\frac{1}{2}(25 + 144 + 169)$

$= -169$ 1

[CBSE Marking Scheme 2012]

Q. 10. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = 5$ and each one of them is perpendicular to the sum of the other two, then find $|\vec{a} + \vec{b} + \vec{c}|$. R&U **[O.D. Comptt. 2011, 2010]**

[O.D. Set I, II, III Comptt. 2013]

Sol. Since $\vec{a} \perp (\vec{b} + \vec{c})$, therefore

$$\vec{a}.(\vec{b} + \vec{c}) = 0$$

or $\vec{a}.\vec{b} + \vec{a}.\vec{c} = 0$

or $\vec{a}.\vec{a}+(\vec{a}.\vec{b}+\vec{a}.\vec{c}) = \vec{a}.\vec{a}+0$ ½

or $\vec{a}.(\vec{a}+\vec{b}+\vec{c}) = |\vec{a}|^2$

or $\vec{a}.(\vec{a}+\vec{b}+\vec{c}) = 3^2 = 9$ ...(i) ½

Similarly, $\vec{b}.(\vec{a}+\vec{b}+\vec{c}) = |\vec{b}|^2 = 16$ ...(ii) ½

and $\vec{c}.(\vec{a}+\vec{b}+\vec{c}) = |\vec{c}|^2 = 25$ ...(iii) ½

Adding eqn. (i), (ii) and (iii), we get

$\vec{a}.(\vec{a}+\vec{b}+\vec{c})+\vec{b}.(\vec{a}+\vec{b}+\vec{c})$

$+\vec{c}.(\vec{a}+\vec{b}+\vec{c}) = 50$ ½

or $(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) = 50$ ½

or $|\vec{a}+\vec{b}+\vec{c}|^2 = 50$ ½

or $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}$ ½

[CBSE Marking Scheme 2013]

AI Q. 11. Find the angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$.

R&U **[NCERT Exemplar]**

[Outside Delhi Set-II, 2015]

Sol. $\vec{a} = \hat{i}-\hat{j} \Rightarrow |\vec{a}| = \sqrt{2}$

$\vec{b} = \hat{j}-\hat{k} \Rightarrow |\vec{b}| = \sqrt{2}$

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\left(\hat{i}-\hat{j}\right).\left(\hat{j}-\hat{k}\right) = \sqrt{2}\times\sqrt{2} \cos\theta$

$-\frac{1}{2} = \cos\theta$

$\theta = \frac{2\pi}{3}$

$\therefore$ angle between vectors is $\frac{2\pi}{3}$

[CBSE Marking Scheme 2015]

Q .12. If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$.

A **[Delhi 2017]**

Sol. $|\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\vec{a}.\vec{b} = 0 = \vec{b}.\vec{c} = \vec{c}.\vec{a}$ ...(i) **1**

Let α, β and γ be the angles made by $(\vec{a}+\vec{b}+\vec{c})$ with $\vec{a},\vec{b}$ and $\vec{c}$ respectively

$(\vec{a}+\vec{b}+\vec{c}).\vec{a} = |\vec{a}+\vec{b}+\vec{c}||\vec{a}| \cos\alpha$

or $\alpha = \cos^{-1}\left(\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$

Similarly, $\beta = \cos^{-1}\left(\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$ and

$\gamma = \cos^{-1}\left(\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$ **1**

using (i), we get α = β = γ ½

Now

$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

1

or $|\vec{a}+\vec{b}+\vec{c}|^2 = 3|\vec{a}|^2$ (using (i))

or $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}|\vec{a}|$

$\therefore$ $= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) = \beta = \gamma$ ½

[CBSE Marking Scheme, 2017]

Q. 13. If $\vec{a},\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, find the angles which the vector $2\vec{a}+\vec{b}+2\vec{c}$ makes with the vectors $\vec{a},\vec{b}$ and $\vec{c}$.

A **[O.D. Comptt. 2017]**

Sol. Let the vector $\vec{P} = (2\vec{a}+\vec{b}+2\vec{c})$ makes angles α, β, γ respectively with the vector $\vec{a},\vec{b},\vec{c}$

Given that $|\vec{a}| = |\vec{b}| = |\vec{c}|$ and $\vec{a}.\vec{b} = \vec{c}.\vec{a} = 0$

1

$\cos\alpha = \frac{(2\vec{a}+\vec{b}+2\vec{c}).\vec{a}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{a}|}$ ½

$= \frac{2|\vec{a}|^2}{3|\vec{a}||\vec{a}|} = \frac{2}{3}$ or $\alpha = \cos^{-1}\frac{2}{3}$ **1**

$\cos\beta = \frac{(2\vec{a}+\vec{b}+2\vec{c}).\vec{b}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{b}|} = \frac{|\vec{b}|^2}{3|\vec{b}||\vec{b}|} = \frac{1}{3}$

or $\beta = \cos^{-1}\frac{1}{3}$ **1**

$\cos\gamma = \frac{(2\vec{a}+\vec{b}+2\vec{c}).\vec{c}}{|2\vec{a}+\vec{b}+2\vec{c}||\vec{c}|}$

$$= \frac{2|\vec{c}|^2}{3|\vec{c}||\vec{c}|} = \frac{2}{3}$$

or $\gamma = \cos^{-1}\frac{2}{3}$ ½

[CBSE Marking Scheme, 2017]

Q. 14. If $\vec{a}$ and $\vec{b}$ are two vectors, such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$, then find $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$.

R&U **[Delhi 2011]**

Sol. Given, $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}\cdot\vec{b} = 1$

Now, $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$

$= 6\vec{a}\cdot\vec{a} + 21\vec{a}\cdot\vec{b} - 10\vec{b}\cdot\vec{a} - 35\vec{b}\cdot\vec{b}$

$= 6|\vec{a}|^2 + 21\vec{a}\cdot\vec{b} - 10\vec{b}\cdot\vec{a} - 35|\vec{b}|^2$ **1**

$[\because \vec{x}\cdot\vec{x} = |\vec{x}|^2$ and $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

$= 6|\vec{a}|^2 + 11\vec{a}\cdot\vec{b} - 35|\vec{b}|^2$

$= 6(2)^2 + 11(1) - 35(1)^2$

$= 24 + 11 - 35 = 0$ **1**

$[\because |\vec{a}| = 2$ and $|\vec{b}| = 1]$

Hence, $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b}) = 0$ **2**

[CBSE Marking Scheme 2011]

Q. 15. If vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

R&U **[Foreign 2011; All India 2009C]**

Sol. Given, $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$

and $\vec{c} = 3\hat{i} + \hat{j}$

Also, $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$.

$\therefore \quad (\vec{a} + \lambda\vec{b})\cdot\vec{c} = 0$ ...(i) **1**

$[\because$ when $\vec{a} \perp \vec{b}$, then $\vec{a}\cdot\vec{b} = 0]$

Now, $\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k})$

or $\vec{a} + \lambda\vec{b} = \hat{i}(2-\lambda) + \hat{j}(2+2\lambda) + \hat{k}(3+\lambda)$ **1**

Then from Eq. (i), we get

$[\hat{i}(2-\lambda) + \hat{j}(2+2\lambda) + \hat{k}(3+\lambda)]\cdot[3\hat{i} + \hat{j}] = 0$ **1**

or $3(2-\lambda) + 1(2+2\lambda) = 0$

or $8 - \lambda = 0$

$\therefore \quad \lambda = 8$ **1**

Q. 16. If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ, so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors. R&U **[All India 2013]**

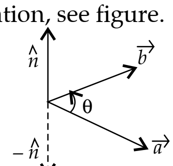
Sol. Try Yourself Like Q.3 LATQ-I.

TOPIC-3
Cross Product

Revision Notes

1. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is defined by,

$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. For better illustration, see figure.



Consider $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

then, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$.

- **Properties/Observations of Cross Product**

  ➲ $\hat{i}\times\hat{i}=|\hat{i}||\hat{i}|\sin 0=\vec{0}$ or $\hat{i}\times\hat{i}=\vec{0}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}$.

  ➲ $\hat{i}\times\hat{j}=|\hat{i}||\hat{j}|\sin\frac{\pi}{2}.\hat{k}=\hat{k}$ or $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$.

  ➲ $\vec{a}\times\vec{b}$ is a vector $\vec{c}$ (say) then this vector $\vec{c}$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.

  ➲ $\vec{a}\times\vec{b}=\vec{0}\Leftrightarrow\vec{a}\,||\,\vec{b}$ or, $\vec{a}=\vec{0},\ \vec{b}=\vec{0}$.

  ➲ $\vec{a}\times\vec{a}=\vec{0}$.

  ➲ $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (Commutative property does not hold for cross product).

  ➲ $\vec{a}\times(\vec{b}+\vec{c})=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (Left distributive).

  ➲ $(\vec{b}+\vec{c})\times\vec{a}=\vec{b}\times\vec{a}+\vec{c}\times\vec{a}$ (Right distributive).

  (Distributive property of the vector product or cross product)

2. Relationship between Vector product and Scalar product [Lagrange's Identity]

or $$|\vec{a}\times\vec{b}|^2+\left(\vec{a}\cdot\vec{b}\right)^2=|\vec{a}|^2\cdot|\vec{b}|^2$$

3. Cauchy-Schwarz inequality :

For any two vectors $\vec{a}$ and $\vec{b}$, always have $|\vec{a}\cdot\vec{b}|\leq|\vec{a}||\vec{b}|$.

Note :

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then the area of triangle can be obtained by evaluating $\frac{1}{2}|\vec{a}\times\vec{b}|$.
- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then the area of parallelogram can be obtained by evaluating $|\vec{a}\times\vec{b}|$.
- The area of the parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a}\times\vec{b}|$.

Know the Formulae

➲ Angle between two vectors $\vec{a}$ and $\vec{b}$ in terms of cross-product can be found by the expression given here :

$$\sin\theta=\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\text{ or }\theta=\sin^{-1}\left(\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right).$$

Theorem

Triangle Inequality

For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}|\leq|\vec{a}|+|\vec{b}|$.

Proof : The given inequality holds trivially when either $\vec{a}=0$ or $\vec{b}=0$ *i.e.*, in such a case $|\vec{a}+\vec{b}|=0=|\vec{a}|+|\vec{b}|$.

So, let us check it for $|\vec{a}|\neq 0\neq|\vec{b}|$.

Then consider $$|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2\,\vec{a}.\vec{b}$$

or $$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta$$

For $\cos\theta \le 1$, we have : $2|\vec{a}||\vec{b}|\cos\theta \le 2|\vec{a}||\vec{b}|$

or $|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|\cos\theta \le |\vec{a}|^2+|\vec{b}|^2+2|\vec{a}||\vec{b}|$

or $$|\vec{a}+\vec{b}|^2 \le \left(|\vec{a}|+|\vec{b}|\right)^2$$

or $$|\vec{a}+\vec{b}| \le |\vec{a}|+|\vec{b}|$$

Hence proved

Objective Type Questions

(1 mark each)

Q. 1. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is

(a) 0 **(b) –1**
(c) 1 **(d) 3**

[NCERT Misc.]

Ans. Correct option : (c)

Explanation :

$\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$

$=\hat{i}.\hat{i}+\hat{j}.(-\hat{j})+\hat{k}.\hat{k}$

$=1-\hat{j}.\hat{j}+1$

$=1-1+1$

$=1$

Q. 2. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to

(a) 0 **(b) π/4**
(c) π/2 **(d) π**

[NCERT Misc.]

Ans. Correct option : (b)

Explanation :

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors, so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

$|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta=|\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow \cos\theta=\sin\theta \quad \left[\because |\vec{a}| \text{ and } |\vec{b}| \text{ are positive.}\right]$

$\Rightarrow \tan\theta=1$

$\Rightarrow \theta=\frac{\pi}{4}$

So that, $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to $\frac{\pi}{4}$.

Q. 3. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(a) π/6 **(b) π/4**
(c) π/3 **(d) π/2**

[NCERT Ex.]

Ans. Correct option : (b)

Explanation :

It is given that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}|=1$.

$|\vec{a}\times\vec{b}|=1$

$\Rightarrow \left||\vec{a}||\vec{b}|\sin\theta\hat{n}\right|=1$

$\Rightarrow |\vec{a}||\vec{b}||\sin\theta|=1$

$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$

$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$

$\Rightarrow \theta=\frac{\pi}{4}$

So that, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Q. 4. The vectors from origin to the points A and B are $\vec{a}=2\hat{i}-3\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$ respectively, then the area of triangle OAB is

(a) 340 **(b) $\sqrt{25}$**
(c) $\sqrt{229}$ **(d) $\frac{1}{2}\sqrt{229}$**

[NCERT Exemp.]

Ans. Correct option : (d)

Explanation :

Area of $\Delta OAB=\frac{1}{2}|\overrightarrow{OA}\times\overrightarrow{OB}|$

$=\frac{1}{2}|(2\hat{i}-3\hat{j}+2\hat{k})\times(2\hat{i}+3\hat{j}+\hat{k})|$

$=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 2\\ 2 & 3 & 1\end{vmatrix}$

$=\frac{1}{2}\left|\left[\hat{i}(-3-6)-\hat{j}(2-4)+\hat{k}(6+6)\right]\right|$

$=\frac{1}{2}|-9i+2\hat{j}+12\hat{k}|$

Area of ΔOAB

$= \frac{1}{2}\sqrt{(81+4+144)}$

$= \frac{1}{2}\sqrt{229}$

Q. 5. For any vector $\vec{a}$, the value of $(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2$ is equal to

(a) $\vec{a}^2$ **(b)** $3\vec{a}^2$

(c) $4\vec{a}^2$ **(d)** $2\vec{a}^2$

[NCERT Exemp.]

Ans. Correct option : (d)

Explanation :

Let $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$

$\therefore \quad \vec{a}^2 = x^2+y^2+z^2$

$$\therefore \quad \vec{a}\times\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix}$$

$= \hat{i}[0]-\hat{j}[-z]+\hat{k}[-y]$

$= z\hat{j}-y\hat{k}$

$\therefore \quad (\vec{a}\times\hat{i})^2 = (z\hat{j}-y\hat{k})(z\hat{j}-y\hat{k})$

$= y^2+z^2$

Similarly, $(\vec{a}\times\hat{j})^2 = x^2+z^2$ and $(\vec{a}\times\hat{k})^2 = x^2+z^2$

$(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2 = y^2+z^2+x^2+z^2+x^2+y^2$

$= 2(x^2+y^2+z^2) = 2\vec{a}^2$

Q. 6. If $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$, then the value of $|\vec{a}\times\vec{b}|$ is

(a) 5 **(b)** 10

(c) 14 **(d)** 16

[NCERT Exemp.]

Ans. Correct option : (d)

Explanation :

Here, $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$ [Given]

$\therefore \quad \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$12 = 10\times 2\cos\theta$

$\Rightarrow \quad \cos\theta = \frac{12}{20} = \frac{3}{5}$

$\Rightarrow \quad \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}}$

$\sin\theta = \pm\frac{4}{5}$

$\therefore \quad |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$

$= 10\times 2\times\frac{4}{5}$

$= 16$

Q. 7. The vectors $\lambda\hat{i}+\hat{j}+2\hat{k}$, $\hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar, if

(a) $\lambda = -2$ **(b)** $\lambda = 0$

(c) $\lambda = 1$ **(d)** $\lambda = -1$

[NCERT Exemp.]

Ans. Correct option : (a)

Explanation :

Let $\vec{a} = \lambda\hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = \hat{i}+\lambda\hat{j}-\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+\lambda\hat{k}$

For $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar,

$$\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda(\lambda^2-1)-1(\lambda+2)+2(-1-2\lambda) = 0$

$\Rightarrow \quad \lambda^3-\lambda-\lambda-2-2-4\lambda = 0$

$\Rightarrow \quad \lambda^3-6\lambda-4 = 0$

$\Rightarrow \quad (\lambda+2)\,(\lambda^2-2\lambda-2) = 0$

$\Rightarrow \quad \lambda = -2$ or $\lambda = \frac{2\pm\sqrt{12}}{2}$

$\Rightarrow \quad \lambda = -2$ or $\lambda = \frac{2\pm 2\sqrt{3}}{2} = 1\pm\sqrt{3}$

Very Short Answer Type Questions

(1 mark each)

Q. 1. Write the value of $(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}$.

R&U [O.D. Set I, 2012]

Sol. $(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}. = \hat{k}.\hat{k}+\hat{i}.\hat{j}$

$= 1 + 0 = 1$ **1**

[CBSE Marking Scheme 2012]

Q. 2. Write the value of $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k}$.

R&U [O.D. Set II, 2012]

Sol. $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k} = -\hat{i}.\hat{i}+0$

$= -\hat{i}.\hat{i} + 0$

$= -1 + 0 = -1$ **1**

[CBSE Marking Scheme 2012]

Q. 3. Write the value of $(\hat{k}\times\hat{i})\cdot\hat{j}+\hat{i}.\hat{k}$.

R&U [O.D. Set III, 2012]

Sol. $(\hat{k}\times\hat{i})\cdot\hat{j}+\hat{i}.\hat{k} = \hat{j}.\hat{j}+\hat{i}.\hat{k}$

$= 1 + 0 = 1$ **1**

[CBSE Marking Scheme 2012]

AI Q. 4. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 3 and $\frac{2}{3}$ respectively such that $\vec{a}\times\vec{b}$ is a unit vector, write the angle between $\vec{a}$ & $\vec{b}$.

R&U [O.D. 2010] [Delhi Set II, 2014] [S.Q.P. 2012]

Sol. We know

$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta.\hat{n}$, where $\hat{n} = 1$

$$|\vec{a} \times \vec{b}| = 1$$

$$|\vec{a}| = 3 \quad \text{[Given]}$$

and $$|\vec{b}| = \frac{2}{3}$$

$\therefore$ $$1 = 3 \times \frac{2}{3} \sin\theta$$

or $$\sin\theta = \frac{1}{2}$$

or $$\theta = \frac{\pi}{6}$$ **1**

[CBSE Marking Scheme 2012]

Q. 5. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector. Write the angle between $\vec{a}$ & $\vec{b}$. R&U **[Delhi Set II, 2014]**

Sol. Since $\vec{a} \times \vec{b}$ is a unit vector, therefore

$$|\vec{a} \times \vec{b}| = 1$$

or $$|\vec{a}||\vec{b}|\sin\theta = 1$$

or $$(\sqrt{3})\left(\frac{2}{3}\right)\sin\theta = 1$$

or $$\sin\theta = \frac{\sqrt{3}}{2}$$

$\therefore$ $$\theta = \frac{\pi}{3}.$$ **1**

[CBSE Marking Scheme 2014]

Q. 6. If $|\vec{a}| = 8, |\vec{b}| = 3$ and $|\vec{a} \times \vec{b}| = 12$, find the angle between $\vec{a}$ and $\vec{b}$.

R&U **[O.D. Set I, II, III Comptt. 2014]**

Sol. We know

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

or $$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{12}{8 \times 3} = \frac{1}{2}$$

$\therefore$ $$\theta = \frac{\pi}{6}$$ **1**

[CBSE Marking Scheme 2014]

Q. 7. For vector $\vec{a}$, if $|\vec{a}| = a$, then write the value of :

$$\left(\vec{a} \times \hat{i}\right)^2 + \left(\vec{a} \times \hat{j}\right)^2 + \left(\vec{a} \times \hat{k}\right)^2.$$

R&U **[NCERT Exemplar]**
[Delhi Set I, II, III Comptt. 2016]

Sol. Let $$\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$$

then $x^2 + y^2 + z^2 = a^2$ (as, $|\vec{a}| = a$)

$$\vec{a} \times \hat{i} = -y\hat{k} + z\hat{j},$$

$$\vec{a} \times \hat{j} = x\hat{k} - z\hat{i},$$

and $$\vec{a} \times \hat{k} = -x\hat{j} + y\hat{i}$$

$$\therefore (\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$$

$$= 2(x^2 + y^2 + z^2) = 2a^2$$ **1**

[CBSE Marking Scheme 2016]

Q. 8. Find the direction cosines of the vector joining the points $A(1, 2, -3)$ and $B(-1, -2, 1)$ directed from B to A.

R&U **[Outside Delhi Set I, II, III Comptt. 2016]**

Sol. $$\vec{BA} = 2\hat{i} + 4\hat{j} - 4\hat{k}$$

or d-ratios of $\vec{BA}$ are 2, 4, – 4

$\therefore$ Direction cosines are : $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$ **1**

[CBSE Marking Scheme 2016]

Q. 9. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ having the same length $\sqrt{2}$ and their vector product is $-\hat{i} - \hat{j} + \hat{k}$.

R&U **[Outside Delhi Set I, II, III Comptt. 2016]**

Sol. $$\sin\theta = \frac{|-\hat{i} - \hat{j} + \hat{k}|}{\sqrt{2}\cdot\sqrt{2}}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

or $$\theta = 60°$$

or $$\theta = \frac{\pi}{3}$$ **1**

[CBSE Marking Scheme 2016]

Q. 10. Find λ and μ if $(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = 0$.

R&U **[O.D. 2016]**

Sol. Getting $\lambda = -9$

and $\mu = 27$ **1**

[CBSE Marking Scheme 2016]

Detailed Solution :

$$(\hat{i} + 3\hat{j} + 9\hat{k}) \times (3\hat{i} - \lambda\hat{j} + \mu\hat{k}) = 0$$

or $$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 9 \\ 3 & -\lambda & \mu \end{vmatrix} = 0$$

or $\hat{i}(3\mu+9\lambda)-\hat{j}(\mu-27)+\hat{k}(-\lambda-9) = 0$

or $3\mu + 9\lambda = 0$ ...(i)

or $\mu - 27 = 0$ ...(ii)

or $-\lambda - 9 = 0$ ...(iii)

from eqn. (ii) and (iii),

$\mu = 27$

and $\lambda = -9$

Q. 11. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $|\vec{a}\times\vec{b}| = \vec{a}\cdot\vec{b}$, then find the angle between $\vec{a}$ and $\vec{b}$. R&U [O.D. 2010] [S.Q.P. 2016]

Sol. $\sin\theta = \cos\theta$

or $\theta = 45°$ 1

[CBSE Marking Scheme 2016]

Q. 12. If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \frac{1}{2}, |\vec{b}| = \frac{4}{\sqrt{3}}$ and $|\vec{a}\times\vec{b}| = \frac{1}{\sqrt{3}}$, then find $|\vec{a}\cdot\vec{a}|$. R&U [O.D. Set II 2016]

Sol. $|\bar{a}| = \frac{1}{2}$ $|\bar{b}| = \frac{4}{\sqrt{3}}$ $|\bar{a}\times\bar{b}| = \frac{1}{\sqrt{3}}$ $|\bar{a}\cdot\bar{b}| = ?$

$|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$ (θ: Angle between vectors $\bar{a}$ & $\bar{b}$)

$\frac{1}{\sqrt{3}} = \frac{1}{2}\times\frac{4}{\sqrt{3}}\sin\theta$

$\sin\theta = \frac{1}{2}$ $\theta = 30°$

$|\bar{a}\cdot\bar{b}| = |\bar{a}||\bar{b}|\cos\theta$

$= \frac{1}{2}\times\frac{4}{\sqrt{3}}\times\frac{\sqrt{3}}{2} = 1$

$\therefore |\bar{a}\cdot\bar{b}| = 1$

1

[Topper's Answer 2016]

Q. 13. Write the angle between the vectors $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$. R&U [Delhi Comptt. 2017]

Sol. Angle between $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ is π. 1

[CBSE Marking Scheme 2017]

Q. 14. If $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 225$ and $|\vec{a}| = 5$, then write the value of $|\vec{b}|$. R&U [O.D. Comptt. 2017]

Sol. $(\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = 225$

or $|\vec{a}|^2|\vec{b}|^2(\sin^2\theta + \cos^2\theta) = 225$

or $(5)^2|\vec{b}|^2 = 225$ or $|\vec{b}| = 3$ 1

[CBSE Marking Scheme 2017]

Q. 15. Write the value of the following.

$\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$

R&U [Foreign 2014]

Sol. $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$

$= \hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{k}+\hat{j}\times\hat{i}+\hat{k}\times\hat{i}+\hat{k}\times\hat{j}$

$= \hat{k}-\hat{j}+\hat{i}-\hat{k}+\hat{j}-\hat{i} = \vec{0}$

$[\because \hat{i}\times\hat{j} = \hat{k}, \hat{i}\times\hat{k} = -\hat{j}, \hat{j}\times\hat{k} = \hat{i}, \hat{j}\times\hat{i} = -\hat{k}\times\hat{i} = \hat{j}, \hat{k}\times\hat{j} = -\hat{i}]$ 1

[CBSE Marking Scheme 2014]

Answering Tip

- Practice of calculation of vectors should be done properly.

Q. 16. Find the angle between $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively, when $|\vec{a}\times\vec{b}| = \sqrt{3}$.

R&U [Delhi 2009]

Sol. Given, $|\vec{a}| = 1, |\vec{b}| = 2$ and $|\vec{a}\times\vec{b}| = \sqrt{3}$

or $|\vec{a}||\vec{b}|\sin\theta = \sqrt{3}$

$[\because \vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta.\hat{n} \text{ and } |\hat{n}| = 1]$

or $1 \times 2 \times \sin\theta = \sqrt{3}$

or $\sin\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$ or $\theta = \frac{\pi}{3}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$. **1**

[CBSE Marking Scheme 2009]

Q. 17. Write the value of *p*, for which $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$ are parallel vectors.

R&U **[Delhi 2009]**

Sol. Given vectors are $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}$ and

$\vec{b} = \hat{i}+p\hat{j}+3\hat{k}$

Also $\vec{a}$ and $\vec{b}$ are parallel vectors.

So, $\vec{a}\times\vec{b} = 0$

or $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 9 \\ 1 & p & 3 \end{vmatrix} = \vec{0}$

or $\hat{i}(6-9p)-\hat{j}(9-9)+\hat{k}(3p-2) = \vec{0}$

or $\hat{i}(6-9p)+\hat{k}(3p-2) = 0\hat{i}+0\hat{j}+0\hat{k}$

On comparing the coefficients of $\hat{i}$ or $\hat{k}$ form both sides, we get

$6-9p = 0$

$\therefore \quad p = \frac{2}{3}$ **1**

[CBSE Marking Scheme 2009]

Alternate Method :

Since $\vec{a}$ and $\vec{b}$ are parallel, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\Rightarrow \frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow p = \frac{2}{3}$.

Q. 18. If $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 400$ and $|\vec{a}| = 5$, then write the value of $|\vec{b}|$. R&U **[Foreign 2016]**

Sol. Try Yourself

Q. 19. Find λ, if

$$(2\hat{i}+6\hat{j}+14\hat{k})\times(\hat{i}-\lambda\hat{j}+7\hat{k}) = 0$$

R&U **[All India 2010]**

Sol. Try Yourself

Q. 20. Find the value of *p*,

if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+3\hat{j}+p\hat{k}) = \vec{0}$

R&U **[All India 2009]**

Sol. Try Yourself

Commonly Made Error

- Some candidates make mistakes while calculating dot & cross product as vector *i.i* = 1 and *i* × *i* = 0 is right method but candidates like *i* × *i* = 1 and *i.i* = 0 which is wrong.

Short Answer Type Questions

(2 marks each)

Q. 1. Find $|\vec{a}\times\vec{b}|$, if $\vec{a} = \hat{i}+2\hat{j}-\hat{k}$, $\vec{b} = 3\hat{i}+\hat{j}-\hat{k}$. R&U

Sol. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & 1 & -1 \end{vmatrix}$

$= \hat{i}(-2+1)-\hat{j}(-1+3)+\hat{k}(1-6)$

$= -i-2j-5k$ **1**

$\left|\vec{a}\times\vec{b}\right| = \sqrt{1^2+2^2+5^2}$

$= \sqrt{1+4+25} = \sqrt{30}$ **1**

Q. 2. Find the area of parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = 2\hat{i}-\hat{j}-\hat{k}$. R&U

Sol. $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & -1 & -1 \end{vmatrix}$

$= i(1+2)-j(-1-4)+k(-1+2)$

$= 3i+5j+k$ **1**

Area of $|\vec{a}\times\vec{b}| = \sqrt{9+25+1} = \sqrt{35}$ sq. units **1**

Q. 3. Find $|\vec{a}\times\vec{b}|$ if $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 12$. R&U

Sol. $\because \quad \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a||b|} = \frac{12}{10\times2} = \frac{3}{5}$

$\cos\theta = \frac{3}{5}$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{9}{25}} = \frac{4}{5} \qquad \mathbf{1}$$

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

or $\quad |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

or $\quad |\vec{a}\times\vec{b}| = 10\times 2\times\frac{4}{5} = 16 \qquad \mathbf{1}$

Q. 4. Using vectors, find the area of triangle ABC, with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

R&U **[Foreign 2017]**

Sol. $\vec{AB} = \hat{i}-3\hat{j}+\hat{k}, \quad \vec{AC} = 3\hat{i}+3\hat{j}-4\hat{k}$ ½+½

$$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$$

$$= \frac{1}{2}\text{ magnitude of }\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix} \qquad \mathbf{1}$$

$$= \frac{\sqrt{274}}{2}\text{ sq. units}$$

[CBSE Marking Scheme 2017]

Answering Tip

- Learn the concept of area of triangle in terms of vector algebra.

Q. 5. If $\vec{a}\times\vec{b} = \vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c} = \vec{b}\times\vec{d}$, then show that $(\vec{a}-\vec{d})$ is parallel to $(\vec{b}-\vec{c})$, it is being given that $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$. **[Foreign 2016] [Delhi 2009]**

Sol. Given,

$$\vec{a}\times\vec{b} = \vec{c}\times\vec{d} \text{ and } \vec{a}\times\vec{c} = \vec{b}\times\vec{d}$$

or $\quad \vec{a}\times\vec{b}-\vec{a}\times\vec{c} = \vec{c}\times\vec{d}-\vec{b}\times\vec{d}$

or $\quad \vec{a}\times(\vec{b}-\vec{c})+(\vec{b}-\vec{c})\times\vec{d} = \vec{0} \qquad \mathbf{1}$

[By left and right distributive law]

or $\quad \vec{a}\times(\vec{b}-\vec{c})-\vec{d}\times(\vec{b}-\vec{c}) = \vec{0} \quad [\because \vec{a}\times\vec{b} = -\vec{b}\times\vec{a}]$

or $\quad (\vec{a}-\vec{d})\times(\vec{b}-\vec{c}) = \vec{0}$

or $\quad (\vec{a}-\vec{d}) \parallel (\vec{b}-\vec{c}) \qquad \mathbf{1}$

[CBSE Marking Scheme 2009]

Long Answer Type Questions-I

(4 marks each)

Q. 1. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram.

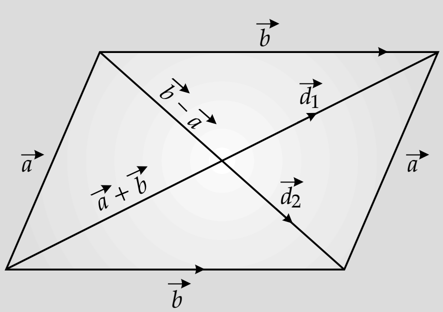
R&U **[O.D. Set I, II, III 2016]**

Sol. $\vec{a} = 2\hat{i}-4\hat{j}-5\hat{k}$

and $\quad \vec{b} = 2\hat{i}+2\hat{j}+3\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = 4\hat{i}-2\hat{j}-2\hat{k}$

and $\quad \vec{b}-\vec{a} = 6\hat{j}+8\hat{k} \qquad$ ½



Unit vector parallel to $\vec{d_1} = \vec{a}+\vec{b}$ is

$$= \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} \qquad ½$$

$$= \frac{4\hat{i}-2\hat{j}-2\hat{j}}{\sqrt{16+4+4}}$$

$$= \frac{4}{\sqrt{24}}\hat{i}-\frac{2}{\sqrt{24}}\hat{j}-\frac{2}{\sqrt{24}}\hat{k}$$

$$= \frac{2}{\sqrt{6}}\hat{i}-\frac{1}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k} \qquad ½$$

Unit vector parallel to $\vec{d_2} = \vec{b}-\vec{a}$ is

$$= \frac{\vec{b}-\vec{a}}{|\vec{b}-\vec{a}|} = \frac{6\hat{j}+8\hat{k}}{\sqrt{36+64}}$$

$$= \frac{6}{10}\hat{j}+\frac{8}{10}\hat{k} = \frac{3}{5}\hat{j}+\frac{4}{5}\hat{k} \qquad ½$$

Area of parallelogram $\frac{1}{2}|\vec{d_1}\times\vec{d_2}|$

$$\therefore \quad \vec{d_1}\times\vec{d_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -2 & -2\\ 0 & 6 & 8\end{vmatrix}$$

$$= \hat{i}(-16+12) - \hat{j}(32-0) + \hat{k}(24-0)$$

$$= -4\hat{i} - 32\hat{j} + 24\hat{k}$$

$$|\vec{d_1} \times \vec{d_2}| = \sqrt{16+1024+576}$$

$$= \sqrt{1,616} \quad ½$$

∴ Area of parallelogram

$$= \frac{1}{2}|\vec{d_1} \times \vec{d_2}| \quad ½$$

$$= \frac{1}{2}\sqrt{1616}$$

$$= \frac{1}{2} \times 4\sqrt{101}$$

$$= 2\sqrt{101}$$

= 20.09 or 20.1 sq. units **1**

[CBSE Marking Scheme 2016]

Q. 2. **If** $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ **find** $(\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$ **.**

R&U [Delhi, 2015]

Sol. $\vec{r} \times \hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}) \times \hat{i}$

$$= -y\hat{k} + z\hat{j} \quad 1$$

$$\vec{r} \times \hat{j} = (x\hat{i} + y\hat{j} + z\hat{k}) \times \hat{j}$$

$$= x\hat{k} - z\hat{i} \quad 1$$

$$\therefore (\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j})$$

$$= (0\hat{i} + z\hat{j} - y\hat{k}) \cdot (-z\hat{i} + 0\hat{j} + x\hat{k})$$

$$= -xy \quad 1$$

$$\therefore (\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$$

$$= -xy + xy = 0 \quad 1$$

[CBSE Marking Scheme 2015]

Q. 3. **If** $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$, $\vec{c} = 2\hat{j} - \hat{k}$ **are three vectors, find the area of the parallelogram having diagonals** $\vec{a} + \vec{b}$ **and** $\vec{b} + \vec{c}$**.**

R&U [Delhi Set I Comptt. 2014]

Sol. $\vec{a} + \vec{b} = \hat{i} - 3\hat{j} + 2\hat{k};\ \vec{b} + \vec{c} = -\hat{i} + 2\hat{j}$ **1**

$$(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{vmatrix} = -4\hat{i} + 2\hat{j} - \hat{k} \quad 1½$$

Area of parallelogram

$$= \frac{1}{2}\left|(\vec{a} + \vec{b}) \times (\vec{b} + \vec{c})\right|$$

$$= \frac{\sqrt{21}}{2} \text{ sq. units} \quad 1½$$

[CBSE Marking Scheme 2014]

Q. 4. **Find the unit vector perpendicular to both the vectors** $\vec{a} + \vec{b}$ **and** $\vec{a} - \vec{b}$**, where** $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ **and** $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$**.**

R&U [Foreign Set I, II, III 2014]

Sol. $\vec{a} + \vec{b} = \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right),$ **1**

and $\vec{a} - \vec{b} = -\hat{j} - 2\hat{k}$ ½

Let $\vec{c} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ **1**

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} \quad ½$$

or $\vec{c} = -2\hat{i} + 4\hat{j} - 2\hat{k}$

or $\hat{c} = -\frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$ **1**

[CBSE Marking Scheme 2014]

Q. 5. **If** $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ **and** $\vec{b} = \hat{j} - \hat{k}$**, find a vector** $\vec{c}$**, such that** $\vec{a} \times \vec{c} = \vec{b}$ **and** $\vec{a} \cdot \vec{c} = 3$**.**

A [Delhi Set II, 2013]

Sol. Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

Given $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$$\vec{b} = \hat{j} - \hat{k}$$

According to the question,

$$\vec{a} \cdot \vec{c} = 3$$

or $(\hat{i} + \hat{j} + \hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = 3$ **1**

or $x + y + z = 3$ ...(i)

and $\vec{a} \times \vec{c} = \vec{b}$

or $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \vec{b} = \hat{j} - \hat{k}$ **1**

or $(z-y)\hat{i} + (x-z)\hat{j} + (y-x)\hat{k} = \hat{j} - \hat{k}$

On equating the coefficients of like terms, we get

$z - y = 0$, or $y = z$ ...(ii)

$x - z = 1$ ...(iii)

and $y - x = -1$ ...(iv) **1**

Solving eqns. (i), (ii), (iii) and (iv), we get

$x = 5/3$

and $y = 2/3 = z$

Hence, $\vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$ **1**

[CBSE Marking Scheme 2013]

AI **Q. 6. Using vectors, find the area of the triangle whose vertices are $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.** R&U **[Delhi 2017] [Delhi Set III, 2013]**

OR

Using vectors find the area of triangle ABC with vertices $A(1, 2, 3)$ $B(2, -1, 4)$ and $C(4, 5, -1)$. R&U **[Delhi 2017]**

Sol. Given, $\vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{OB} = 2\hat{i} - \hat{j} + 4\hat{k}$

and $\vec{OC} = 4\hat{i} + 5\hat{j} - \hat{k}$

Now, $\vec{AB} = \vec{OB} - \vec{OA}$

$= (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= \hat{i} - 3\hat{j} + \hat{k}$ **½**

and $\vec{AC} = \vec{OC} - \vec{OA}$

$= (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 3\hat{i} + 3\hat{j} - 4\hat{k}$ **½**

$\therefore$ The area of the given triangle

$= \frac{1}{2}|\vec{AB} \times \vec{AC}|$ **1**

Now, $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$

$= \hat{i}(12-3) + \hat{j}(3+4) + \hat{k}(3+9)$

$= 9\hat{i} + 7\hat{j} + 12\hat{k}$ **1**

Therefore,

$|\vec{AB} \times \vec{AC}| = \sqrt{(9)^2 + (7)^2 + (12)^2}$

$= \sqrt{81 + 49 + 144}$

$= \sqrt{274}$

Hence,

required area $= \frac{1}{2}\sqrt{274}$. unit^2 **1**

[CBSE Marking Scheme 2013]

Q. 7. Find the unit vector perpendicular to the plane of ΔABC whose vertices are A (3, – 1, 2), B (1, – 1, – 3) and C (4, – 3, 1) R&U **[S.Q.P., 2013]**

Sol. A vector $\perp_r$ to the plane of ΔABC,

$\vec{OA} = 3\hat{i} - \hat{j} + 2\hat{k}$,

$\vec{OB} = \hat{i} - \hat{j} - 3\hat{k}$,

$\vec{OC} = 4\hat{i} - 3\hat{j} + \hat{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\hat{i} + 0\hat{j} - 5\hat{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 3\hat{i} - 2\hat{j} + 4\hat{k}$

Unit vector perpendicular to plane $= \dfrac{\vec{AB} \times \vec{BC}}{|\vec{AB} \times \vec{BC}|}$

$\vec{AB} \times \vec{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 3 & -2 & 4 \end{vmatrix} = -10\hat{i} + 7\hat{j} + 4\hat{k}$ **1**

$|\vec{AB} \times \vec{BC}| = \sqrt{100 + 49 + 16} = \sqrt{165}$ **1**

$\therefore$ Unit vector $\perp_r$ to the plane

$= \frac{1}{\sqrt{165}}(-10\hat{i} + 7\hat{j} + 4\hat{k})$. **1**

[CBSE Marking Scheme 2013]

Q. 8. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p} \cdot \vec{c} = 18$. R&U **[O.D. Set I, II, III, 2012]**

Sol. $\vec{p}$ is $\perp$ to both $\vec{a}$ and $\vec{b}$

or $\vec{p} = \lambda(\vec{a} \times \vec{b})$ **1½**

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$

$$= 32\hat{i} - \hat{j} - 14\hat{k}$$ **1**

Given that $\vec{p}\cdot\vec{c} = 18$

or $\lambda(32\hat{i} - \hat{j} - 14\hat{k})(2\hat{i} - \hat{j} + 4\hat{k}) = 18$ **1**

or $\lambda(64 + 1 - 56) = 18$

$\lambda = 2$

$\therefore \quad \vec{p} = 64\hat{i} - 2\hat{j} - 28\hat{k}.$ ½

[CBSE Marking Scheme 2012]

Q. 9. If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, then find $|\vec{a}\times\vec{b}|$. R&U **[Outside Delhi Set-II, 2015]**

Sol.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3)$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2}$$

$$= \sqrt{289 + 169 + 49}$$

$$= \sqrt{507}$$

$$= \sqrt{3\times 169}$$

$$|\vec{a}\times\vec{b}| = 13\sqrt{3}$$

[CBSE Marking Scheme 2015]

Q. 10. If $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{b}$ in the form of $\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1 \,||\, \vec{a}$ and $\vec{b}_2$ perpendicular to $\vec{a}$.

A **[NCERT] [O.D. Set I, II, III, 2013]**

Sol. Here $\vec{a} = 3\hat{i} - \hat{j}$

and $\vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

To express $\vec{b} = \vec{b}_1 + \vec{b}_2$,

where $\vec{b}_1 \parallel \vec{a}$ or $\vec{b}_1 = \lambda\vec{a}$

$\vec{b}_1 = \lambda(3\hat{i} - \hat{j})$ ½

Let $b_2 = x\hat{i} + y\hat{j} + z\hat{k}$

Now, $\vec{b}_2 \perp \vec{a}$ or $\vec{b}_2\cdot\vec{a}$ ½

$3x - y = 0$ ...(i)

Now, $\vec{b} = \vec{b}_1 + \vec{b}_2$

or $2\hat{i} + \hat{j} - 3\hat{k} = (3\lambda + x)\hat{i} + (y - \lambda)j + z\hat{k}$

Comparing the corresponding components

$2 = 3\lambda + x$ ...(ii) ½

$1 = -\lambda + y$

or $\lambda = y - 1$ ...(iii) ½

$-3 = z$ ...(iv) ½

From eqn. (ii), $2 = 3(y - 1) + x$

or $2 = 3y - 3 + x$

or $x + 3y = 5$ ...(v) ½

Solving eqn. (i) & (v), $x = \dfrac{1}{2}$, $y = \dfrac{3}{2}$

$\therefore$ From eqn. (iii), $\lambda = \dfrac{1}{2}$

$\therefore \quad \vec{b}_1 = \dfrac{3}{2}\hat{i} - \dfrac{1}{2}\hat{j}$ ½

and $\vec{b}_2 = \dfrac{1}{2}\hat{i} + \dfrac{3}{2}\hat{j} - 3\hat{k}$ ½

[CBSE Marking Scheme 2013]

Commonly Made Error

- Generally students do not able to find the vector $\vec{b}$ in the form of $\vec{b} + \vec{b}_2$, instead they add vector $\vec{a}$ and $\vec{b}$ which leads incorrect result.

Answering Tip

- Read and understand the question carefully to avoid such errors.

Q. 11. Find a unit vector perpendicular to both of the vectors $3\vec{a} + 2\vec{b}$ and $3\vec{a} - 2\vec{b}$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

R&U **[Delhi Set I, II, III, Comptt. 2016]**

Sol. Here $3\vec{a} + 2\vec{b} = 5\hat{i} + 7\hat{j} + 9\hat{k}$

and $3\vec{a} - 2\vec{b} = \hat{i} - \hat{j} - 3\hat{k}$ **1 + 1**

Let $\vec{c}$ be the vector perpendicular to both $(3\vec{a} + 2\vec{b})$ & $(3\vec{a} - 2\vec{b})$.

Then, $\vec{c} = (3\vec{a} + 2\vec{b})\times(3\vec{a} - 2\vec{b})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 7 & 9 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= -12\hat{i} + 24\hat{j} - 12\hat{k}$$ **2**

[CBSE Marking Scheme 2016]

Detailed Solution :

Given $\vec{a} = \hat{i}+\hat{j}+\hat{k}$

$\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$

$3\vec{a} = 3\hat{i}+3\hat{j}+3\hat{k}$

and $2\vec{b} = 2\hat{i}+4\hat{j}+6\hat{k}$ **1**

Let $\vec{c}$ be the vector perpendicular to both $(3\vec{a}+2\vec{b})$ and $(3\vec{a}-2\vec{b})$.

$\therefore \quad 3\vec{a}+2\vec{b} = 5\hat{i}+7\hat{j}+9\hat{k}$ **1**

and $3\vec{a}-2\vec{b} = \hat{i}-\hat{j}-3\hat{k}$ **1**

$$\therefore \quad \vec{c} = (3\vec{a}+2\vec{b})\times(3\vec{a}-2\vec{b})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 7 & 9 \\ 1 & -1 & -3 \end{vmatrix}$$

$$= \hat{i}(-21+9)-\hat{j}(-15-9)+\hat{i}(-5-7)$$

$$= -12\hat{i}+24\hat{j}-12\hat{k}$$ **1**

Q. 12. Show that the points *A*, *B*, *C* with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle, hence find the area of the triangle. R&U **[O.D. Set-I, 2017]**

Sol. $\vec{AB} = -\hat{i}-2\hat{j}-6\hat{k},\ \vec{BC} = 2\hat{i}-\hat{j}+\hat{k},\ \vec{CA} = -\hat{i}+3\hat{j}+5\hat{k}$ **1**

Since $\vec{AB}, \vec{BC}, \vec{CA}$ are not parallel vectors, and $\vec{AB}+\vec{BC}+\vec{CA} = \vec{0}$ $\therefore$ *A*, *B*, *C* form a triangle **1**

Also $\vec{BC}\cdot\vec{CA} = 0$ $\therefore$ *A*, *B*, *C* form a right triangle **1**

Area of $\Delta = \frac{1}{2}|\vec{AB}\times\vec{BC}| = \frac{1}{2}\sqrt{210}$ **[CBSE Marking Scheme 2017] 1**

OR

$\vec{OA} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{OB} = \hat{i}-3\hat{j}-5\hat{k}$

$\vec{OC} = 3\hat{i}-4\hat{j}-4\hat{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = -\hat{i}-2\hat{j}-6\hat{k}$ $\quad |\vec{AB}| = \sqrt{(-1)^2+(-2)^2+(-6)^2} = \sqrt{41}$ unit

$\vec{AC} = \vec{OC}-\vec{OA} = \hat{i}-3\hat{j}-5\hat{k}$ $\quad |\vec{AC}| = \sqrt{1^2+(-3)^2+(-5)^2} = \sqrt{35}$ units

$\vec{BC} = \vec{OC}-\vec{OB} = 2\hat{i}-\hat{j}+\hat{k}$ $\quad |\vec{BC}| = \sqrt{(2)^2+(-1)^2+(1)^2} = \sqrt{6}$ units

$\vec{AC}\cdot\vec{BC} = 2+3-5 = 0$ Hence $\vec{AC}\perp\vec{BC}$ Hence $\angle C = 90°$

$|\vec{BC}|^2+|\vec{AC}|^2 = |\vec{AB}|^2$

Hence ΔABC is right angled at C.

area $= \frac{1}{2}|\vec{AC}\times\vec{BC}|$

$= \frac{1}{2}\left|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -5 \\ 2 & -1 & 1 \end{vmatrix}\right| = \frac{1}{2}|\hat{i}(-8)-\hat{j}(11)+\hat{k}(5)|$

$= \frac{1}{2}\sqrt{(-8)^2+(-11)^2+(5)^2}$

$= \frac{1}{2}\sqrt{210}$ sq. units

$= \sqrt{\frac{210}{4}} = \sqrt{52.5}$ sq. units.

4

[Topper's Answer 2017]

AI Q. 13. If $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$ then express $\vec{b}$ in the form of $\vec{b} = \vec{b_1} + \vec{b_2}$, where $\vec{b_1}$ is parallel to $\vec{a}$ and $\vec{b_2}$ is perpendicular to $\vec{a}$.

R&U **[O.D. 2017]**

Sol. $\vec{b_1} \parallel \vec{a}$ or let $\vec{b_1} = \lambda(2\hat{i} - \hat{j} - 2\hat{k})$ ½

$\vec{b_2} = \vec{b} - \vec{b_1}$

$= (7\hat{i} + 2\hat{j} - 3\hat{k}) - (2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k})$ ½

$= (7 - 2\lambda)\hat{i} + (2 + \lambda)\hat{j} - (3 - 2\lambda)\hat{k}$ **1**

$\vec{b_2} \perp \vec{a}$ or $2(7 - 2\lambda) - 1(2 + \lambda) + 2(3 - 2\lambda) = 0$

or $\lambda = 2$

$\therefore$ $\vec{b_1} = 4\hat{i} - 2\hat{j} - 4\hat{k}$ ½

and $\vec{b_2} = 3\hat{i} + 4\hat{j} + \hat{k}$ ½

or $(7\hat{i} + 2\hat{j} - 3\hat{k}) = (4\hat{i} - 2\hat{j} - 4\hat{k}) + (3\hat{i} + 4\hat{j} + \hat{k})$ **1**

[CBSE Marking Scheme 2017]

Answering Tip

- Clarify the concept of scalar protection of vector thoroughly.

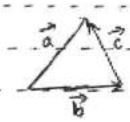
Q. 14. Given that vectors $\vec{a}, \vec{b}, \vec{c}$ form a triangle such that $\vec{a} = \vec{b} + \vec{c}$. Find *p, q, r, s* such that area of triangle is $5\sqrt{6}$ where $\vec{a} = p\hat{i} + q\hat{j} + r\hat{k}, \vec{b} = s\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$.

K&A **[O.D. Set II 2016]**

Sol. $\vec{a} = \vec{b} + \vec{c}$ Here $\vec{a} = p\hat{i} + q\hat{j} + r\hat{k}$

$\vec{b} = s\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{c} = 3\hat{i} + \hat{j} - 2\hat{k}$



$\vec{a} = \vec{b} + \vec{c}$

$\therefore p\hat{i} + q\hat{j} + r\hat{k} = s\hat{i} + 3\hat{j} + 4\hat{k} + 3\hat{i} + \hat{j} - 2\hat{k}$

$p\hat{i} + q\hat{j} + r\hat{k} = (s+3)\hat{i} + 4\hat{j} + 2\hat{k}$

Equating components,

$p = s+3$

$q = 4$

$r = 2$

Area of triangle $= 5\sqrt{6}$

But Area of triangle $= \frac{1}{2}|\vec{a} \times \vec{b}|$

$5\sqrt{6} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ p & 4 & 2 \\ p-3 & 3 & 4 \end{vmatrix} = \frac{1}{2}\left|10\hat{i} - (2p+6)\hat{j} + (12-p)\hat{k}\right|$

$\therefore 600 = 100 + (2p+6)^2 + (12-p)^2$

$500 = 4p^2 + 36 + 24p + 144 + p^2 - 24p \Rightarrow 5p^2 = 320$

$p^2 = 64$

$p = \pm 8$

If $p = 8$, $s = p - 3 = 5$

If $p = -8$, $s = p - 3 = -11$

4

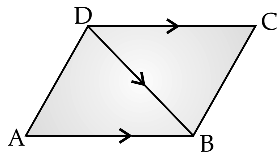
[Topper's Answer 2016]

Q. 15. Find the area of a parallelogram *ABCD* whose side *AB* and the diagonal *DB* are given by the vectors $5\hat{i}+7\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$ respectively.

K **[Foreign 2017]**

Sol. $\vec{AD} = \vec{AB}-\vec{DB} = 3\hat{i}-2\hat{j}+4\hat{k}$ **1**

Area $= |\vec{AB}\times\vec{AD}|$



$= \text{magnitude of} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 7 \\ 3 & -2 & 4 \end{vmatrix}$ **1**

$= |14\hat{i}+\hat{j}-10\hat{k}|$ **1**

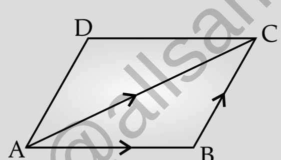
$= \sqrt{297}$ sq. units or $3\sqrt{33}$ sq. units **1**

Q. 16. Find the area of a parallelogram *ABCD* whose side *AB* and the diagonal *AC* are given by the vectors $3\hat{i}+\hat{j}+4\hat{k}$ and $4\hat{i}+5\hat{k}$ respectively.

R&U **[Foreign 2017]**

Sol. $\vec{BC} = \vec{AC}-\vec{AB} = \hat{i}-\hat{j}+\hat{k}$ **1**

Area $= |\vec{AB}\times\vec{BC}|$



$= \text{magnitude of} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$ **1**

$= |5\hat{i}+\hat{j}-4\hat{k}|$ **1**

$= \sqrt{42}$ sq. units **1**

[CBSE Marking Scheme 2017]

Q. 17. If $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$, $\vec{b} = 4\hat{i}-7\hat{j}+\hat{k}$, find a vector $\vec{c}$ such that $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}\cdot\vec{c} = 6$.

K&U **[Foreign 2017]**

Sol. Let $\vec{c} = x\hat{i}+y\hat{j}+z\hat{k}$; $\vec{a}\cdot\vec{c} = 6$ or $2x + y - z = 6$

Now, $\vec{a}\times\vec{c} = \vec{b}$

or $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ x & y & z \end{vmatrix} = 4\hat{i}-7\hat{j}+\hat{k}$ **1½**

or $\hat{i}(z+y)-\hat{j}(2z+x)+\hat{k}(2y-x) = 4\hat{i}-7\hat{j}+\hat{k}$

or $z + y = 4$, $2z + x = 7$, $2y - x = 1$ **1**

Solving and getting $x = 3$, $y = 2$, $z = 2$

$\vec{c} = 3\hat{i}+2\hat{j}+2\hat{k}$ **1½**

[CBSE Marking Scheme 2017]

Q. 18. If $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c} = 3\hat{i}-4\hat{j}-5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$. R&U **[All India 2015]**

Sol. Try Yourself

Q. 19. Find a unit vector perpendicular to both of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. R&U **[Foreign 2014]**

Sol. Try Yourself

Q. 20. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$. R&U **[Delhi 2011]**

Sol. Try Yourself

Q. 21. Using vectors, find the area of triangle with vertices *A*(1, 1, 2), *B*(2, 3, 5) and *C*(1, 5, 5).

R&U **[All India 2011]**

Sol. Try Yourself

Q. 22. Using vectors, find the area of triangle with vertices *A*(2, 3, 5), *B*(3, 5, 8) and *C*(2, 7, 8).

R&U **[Delhi 2010C]**

Sol. Try Yourself

TOPIC-4
Scalar Triple Product

Revision Notes

1. Scalar Triple Product

If $\vec{a}, \vec{b}$ and $\vec{c}$ are any three vectors, then the scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$ is called scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$.

Thus, $(\vec{a} \times \vec{b}).\vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$.

- **Notation for scalar triple product :** The scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$ *i.e.,* $(\vec{a}\times\vec{b}).\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$.
- **Properties/Observations of Scalar Triple Product**
  - ➲ $(\vec{a}\times\vec{b}).\vec{c} = \vec{a}.(\vec{b}\times\vec{c})$. *i.e.,* the position of dot and cross can be interchanged without change in the value of the scalar triple product (provided their cyclic order remains the same).
  - ➲ $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$, *i.e.,* the value of scalar triple product doesn't change when cyclic order of the vectors is maintained.

    Also, $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}]; [\vec{b}\ \vec{c}\ \vec{a}] = -[\vec{b}\ \vec{a}\ \vec{c}]$. *i.e.,* the value of scalar triple product remains the same in magnitude but changes the sign when cyclic order of the vectors is altered.
  - ➲ For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ, we have $[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$.
  - ➲ The value of scalar triple product is zero if any two of the three vectors are identical. That is, $[\vec{a}\ \vec{a}\ \vec{c}] = 0 = [\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}]$ etc.
  - ➲ Value of scalar triple product is zero if any two of the three vectors are parallel or collinear.
  - ➲ Scalar triple product of $\hat{i}, \hat{j}$ and $\hat{k}$ is 1 (unity) *i.e.,* $[\hat{i}\ \hat{j}\ \hat{k}] = 1$
  - ➲ If $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, then the non-parallel and non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are **coplanar.**

Know the Formulae

Volume of Parallelepiped

- If $\vec{a}, \vec{b}$ and $\vec{c}$ represent the three co-terminus edges of a parallelepiped, then its volume can be obtained by : $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b}).\vec{c}$. *i.e.,*

  $(\vec{a}\times\vec{b}).\vec{c}$ = Base area of Parallelepiped × Height of Parallelepiped on this base

Note :

- If for any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, then volume of parallelepiped with the co-terminus edges as $\vec{a}$, b and $\vec{c}$, is zero. This is possible only if the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are co-planar.

Very Short Answer Type Questions (1 mark each)

Q. 1. Find $\vec{a}.(\vec{b}\times\vec{c})$, if

$\vec{a}=2\hat{i}+\hat{j}+3\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and

$\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$. R&U [O.D. Set I, II, III, 2014]

Sol.

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 1\\ 3 & 1 & 2\end{vmatrix}$$

$$=\hat{i}(4-1)-\hat{j}(-2-3)+\hat{k}(-1-6)$$

$$=3\hat{i}+5\hat{j}-7\hat{k}$$

$$\therefore \quad \vec{a}.(\vec{b}\times\vec{c})=(2\hat{i}+\hat{j}+3\hat{k}).(3\hat{i}+5\hat{j}-7\hat{k})$$

$$=6+5-21=-10 \qquad 1$$

[CBSE Marking Scheme 2014]

Q. 2. Find λ, if the vectors

$\vec{a}=\hat{i}+3\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}-\hat{j}-\hat{k}$ and

$\vec{c}=\lambda\hat{j}+3\hat{k}$ are coplanar. R&U [Delhi, 2015]

Sol.

$$\begin{vmatrix}1 & 3 & 1\\ 2 & -1 & -1\\ 0 & \lambda & 3\end{vmatrix}=0 \qquad ½$$

or $\lambda = 7$ ½

[CBSE Marking Scheme 2015]

Short Answer Type Questions (2 marks each)

Q. 1. If the vectors $\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}+\lambda\hat{j}-3\hat{k}$ are coplanar, then find the value of λ.

R&U [O.D. Comptt, 2017]

Sol. For three vectors to be coplanar

$$\begin{vmatrix}1 & -1 & 1\\ 3 & 1 & 2\\ 1 & \lambda & -3\end{vmatrix}=0 \qquad 1$$

or $\lambda = 15$ 1

[CBSE Marking Scheme 2017]

Q. 2. If the vectors $\vec{a}=\hat{i}+3\hat{j}+\hat{k}$ $\vec{b}=\lambda\hat{i}+7\hat{j}+3\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}-\hat{k}$ are coplanar, then find the value of λ. R&U [O.D. Comptt, 2017]

Sol. For coplanarity of vectors

$$\begin{vmatrix}1 & 3 & 1\\ 2 & -1 & -1\\ \lambda & 7 & 3\end{vmatrix}=0 \qquad 1$$

$$4-18+3\lambda+14+\lambda=0$$

Solving to get $\lambda = 0$ 1

[CBSE Marking Scheme 2017]

Long Answer Type Questions-I (4 marks each)

Q. 1. Prove that :

$\vec{a}\cdot\{(\vec{b}+\vec{c})\times(\vec{a}+2\vec{b}+3\vec{c})\}=[\vec{a}\ \vec{b}\ \vec{c}]$.

R&U [S.Q.P. 2016-17]

Sol. LHS $=\vec{a}\cdot(\vec{b}\times\vec{a}+2\vec{b}\times\vec{b}+3\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+2\vec{c}\times\vec{b}+3\vec{c}\times\vec{c})$ 1

$$=\vec{a}\cdot(\vec{b}\times\vec{a})+3\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{c}\times\vec{a})+2\vec{a}\cdot(\vec{c}\times\vec{b})$$

as $\vec{b}\times\vec{b}=\vec{c}\times\vec{c}=0$ 1

$$=3[\vec{a}\ \vec{b}\ \vec{c}]+2[\vec{a}\ \vec{c}\ \vec{b}] \qquad 1$$

$$=3[\vec{a}\ \vec{b}\ \vec{c}]-2[\vec{a}\ \vec{b}\ \vec{c}]$$

$$=[\vec{a}\ \vec{b}\ \vec{c}] \qquad 1$$

[CBSE Marking Scheme 2016]

Commonly Made Error

- Many candidates make errors while simplifying the scalar triple product.

Answering Tip

- Scalar triple product and its applications need to be practiced with the help of practical examples.

Q. 2. Show that the four points $A(4, 5, 1)$, $B(0, -1, -1)$, $C(3, 9, 4)$ and $D(-4, 4, 4)$ are coplanar.

R&U [CBSE Outside Delhi, 2016]

Sol.
$$\vec{AB} = -4\hat{i} - 6\hat{j} - 2\hat{k},$$
$$\vec{AC} = -\hat{i} + 4\hat{j} + 3\hat{k},$$
and $$\vec{AD} = -8\hat{i} - \hat{j} + 3\hat{k}$$ 1½

For 4 points to be coplanar,
$$|\vec{AB}\ \vec{AC}\ \vec{AD}| = 0$$
i.e., $$\begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0$$ 1½

= – 4(12 + 3) + 6(– 3 + 24) – 2(1 + 32)
= – 60 + 126 – 66 = 0, which is true
Hence, points are coplanar. 1

[CBSE Marking Scheme 2016]

Alternative Method :

If four points *A, B, C, D* are coplanar, then vector $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ will be coplanar and so $|\vec{AB}\,\vec{AC}\,\vec{AD}| = 0$

$$A = (4, 5, 1)$$
$$B = (0, -1, -1)$$
$$C = (3, 9, 4)$$
$$D = (-4, 4, 4)$$

By considering *O* = (0, 0, 0) as initial point
$$\vec{OA} = 4\hat{i} + 5\hat{j} + \hat{k}$$
$$\vec{OB} = -\hat{j} - \hat{k},$$
$$\vec{OC} = 3\hat{i} + 9\hat{j} + 4\hat{k}$$
and $$\vec{OD} = -4\hat{i} + 4\hat{j} + 4\hat{k}$$ ½

$\therefore$ $$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= -\hat{j} - \hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -4\hat{i} - 6\hat{j} - 2\hat{k}$$ ½
$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= 3\hat{i} + 9\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -\hat{i} + 4\hat{j} + 3\hat{k}$$ ½
and $$\vec{AD} = \vec{OD} - \vec{OA}$$
$$= -4\hat{i} + 4\hat{j} + 4\hat{k} - (4\hat{i} + 5\hat{j} + \hat{k})$$
$$= -8\hat{i} - \hat{j} + 3\hat{k}$$ ½

Now,
$$|\vec{AB}\ \vec{AC}\ \vec{AD}| = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$
= – 4(12 + 3) + 6(– 3 + 24) + (– 2)(1 + 32)
= – 60 + 126 – 66
= 0 1

$\therefore$ $\vec{AB}, \vec{AC}, \vec{AD}$ are coplanar and these three vectors are co-initial vectors. So, points *A, B, C, D* are coplanar. 1

Q. 3. Prove that
$$[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{d}] = [\vec{a}\ \ \vec{b}\ \ \vec{d}] + [\vec{a}\ \ \vec{c}\ \ \vec{d}].$$

A **[O.D. Set I, II, III Comptt. 2015]**

Sol. Taking LHS $= \vec{a}\cdot\{(\vec{b}+\vec{c})\times\vec{d}\}$ 1 + 1
$$= \vec{a}\cdot\{(\vec{b}\times\vec{d}) + (\vec{c}\times\vec{d})\}$$ 1
$$= \vec{a}\cdot(\vec{b}\times\vec{d}) + \vec{a}\cdot(\vec{c}\times\vec{d})$$ 1
$$= [\vec{a}\ \ \vec{b}\ \ \vec{d}] + [\vec{a}\ \ \vec{c}\ \ \vec{d}]$$

[CBSE Marking Scheme 2015]

AI **Q. 4. If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are coplanar, prove that the vectors $\vec{a}+\vec{b}, \vec{b}+\vec{c}$ and $\vec{c}+\vec{a}$ are also coplanar.** R&U **[Delhi Set I Comptt. 2014]**
[Delhi Set III Comptt. 2013]
[Foreign Set I, II, III, 2014] [Delhi Set I, II, III, 2016]

Sol. Here, $(\vec{a}+\vec{b}), (\vec{b}+\vec{c}), (\vec{c}+\vec{a})$ are coplanar, 1
$$\therefore (\vec{a}+\vec{b})\cdot\left\{(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})\right\} = 0$$ 1
or $$(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}) = 0$$ 1
or $$\vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}(\vec{c}\times\vec{a})$$
$$+ \vec{a}(\vec{c}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{a})$$
$$+ \vec{b}(\vec{c}\times\vec{c}) + \vec{b}\cdot(\vec{c}\times\vec{a}) = 0$$ ½
or $$2\left\{\vec{a}\cdot(\vec{b}\times\vec{c})\right\} = 0$$
$\because$ $$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$$ ½

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

Similarly converse part can also be proved.

[CBSE Marking Scheme 2014]

Q. 5. Find the value of λ, if the points with position vectors $3\hat{i} - 2\hat{j} - \hat{k}, 2\hat{i} + 3\hat{j} - 4\hat{k}, -\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k}$ are coplanar. R&U **[S.Q.P. 2013]**

Sol. Let the points be *A* (3, – 2, – 1), *B* (2, 3, – 4), *C* (– 1, 1, 2) and *D* (4, 5, λ)

$\vec{AB}$ = (Position vector of B) – (Position vector of A)

$\therefore \quad \vec{AB} = -\hat{i}+5\hat{j}-3\hat{k},$

Similarly

$\vec{AC} = -4\hat{i}+3\hat{j}+3\hat{k}$

and $\quad \vec{AD} = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$ **1½**

A, B, C, D are coplanar if $[\vec{AB}\ \vec{AC}\ \vec{AD}]=0$ ½

$$[\vec{AB}\ \vec{AC}\ \vec{AD}] = \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$ ½

$\therefore$ 1(15 + 9) – 7 (– 3 – 12) + (λ + 1) (– 3 + 20) = 0 **1**

24 + 105 + 17λ + 17 = 0

or $\quad \lambda = -\frac{146}{17}$ ½

Commonly Made Error

- Some candidates fail to apply condition of coplanarity.

Answering Tip

- Scalar triple product and its applications need to be practiced with the help of practical examples.

Q. 6. If $\vec{a} = 2\hat{i}-3\hat{j}+4\hat{k}, \vec{b} = \hat{i}+2\hat{j}-3\hat{k}$ and $\vec{c} = 3\hat{i}+4\hat{j}-\hat{k}$, then find $\vec{a}.(\vec{b}\times\vec{c})$ and $(\vec{a}\times\vec{b}).\vec{c}$. Is, $\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$? R&U

Sol. $$\vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ 3 & 4 & -1 \end{vmatrix}$$ ½

$= 2(-2+12)+3(-1+9)+4(4-6)$

= 20 + 24 – 8

= 36 **1**

and $\quad (\vec{a}\times\vec{b}).\vec{c} = \vec{c}.(\vec{a}\times\vec{b})$

$$= \begin{vmatrix} 3 & 4 & -1 \\ 2 & -3 & 4 \\ 1 & 2 & -3 \end{vmatrix}$$ ½

$= 3(9-8)-4(-6-4)-1(4+3)$

= 3 + 40 – 7 = 36 **1**

$\therefore \quad \vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$. **1**

[CBSE Marking Scheme 2013]

Q. 7. If $\vec{p} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{q} = \hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude $5\sqrt{3}$ units perpendicular to the vector $\vec{q}$ and coplanar with vectors $\vec{p}$ and $\vec{q}$. R&U **(SQP 2018-19)**

Sol. Let $\vec{r} = a\hat{i}+b\hat{j}+c\hat{k}$ be the required vector.

Since, $\vec{r}\perp\vec{q}$

therefore, $1a - 2b + 1c = 0$ ...(1) **1**

Also, $\vec{p}, \vec{q}$ and $\vec{r}$ are coplanar.

$\therefore \quad [\vec{p}\ \vec{q}\ \vec{r}] = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ a & b & c \end{vmatrix} = 0 \Rightarrow a-c=0$$ ...(2) **1**

Solving equation (1) and (2)

$$\frac{a}{2-0} = \frac{b}{1+1} = \frac{c}{0+2}$$

$$\Rightarrow \quad \frac{a}{2} = \frac{b}{2} = \frac{c}{2}$$

i.e., $$\frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

$\therefore \quad \vec{r} = 1\hat{i}+1\hat{j}+1\hat{k}$

$|\vec{r}| = \sqrt{3}$

$\therefore$ Unit vector $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ **1**

$\therefore$ Required vector $= 5\sqrt{3}\hat{r} = 5\left(\hat{i}+\hat{j}+\hat{k}\right)$ **1**

[CBSE Marking Scheme 2018-19]

Q. 8. Find x such that the four points A(4, 1, 2), B(5, x, 6), C(5, 1, – 1) and D(7, 4, 0) are coplanar. A **[Outside Delhi Set-II, 2015]**

Sol. Here, $\vec{AB}, \vec{BC}, \vec{CD}$ are coplanar

So $\vec{AB}.\left(\vec{BC}\times\vec{CD}\right) = 0$ **1**

triple product is 0.

$\vec{AB} = 1\hat{i}+(x-1)\hat{j}+4\hat{k}$

$\vec{BC} = 0\hat{i}+(1-x)\hat{j}-7\hat{k}$

$\vec{CD} = 2\hat{i}+3\hat{j}+\hat{k}$ **1½**

$$\vec{AB}.(\vec{BC}\times\vec{CD}) = \begin{vmatrix} 1 & x-1 & 4 \\ 0 & 1-x & -7 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

expanding by R_1

$1(1 - x + 21) - (x - 1)\,14 + 4(2(x - 1) = 0$

$22 - x - 14x + 14 + 8x - 8 = 0$

$-7x = -28$

$x = 4$ 1½

[CBSE Marking Scheme 2015]

AI **Q. 9. If the vector $\vec{p} = a\hat{i} + \hat{j} + \hat{k}$, $\vec{q} = \hat{i} + b\hat{j} + \hat{k}$ and $\vec{r} = \hat{i} + \hat{j} + c\hat{k}$ are coplanar, then for $a, b, c \neq 1$, then show that.**

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

R&U **[Outside Delhi Set I, II, III, Comptt. 2016]**
[SQP Dec. 2016-17]

Sol. Since the vector $\vec{p}, \vec{q}$ and $\vec{r}$ are coplanar,

$\therefore \quad [\vec{p}, \vec{q}, \vec{r}] = 0$ **1**

i.e.,
$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

and $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1,$ **1**

or
$$\begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0$$

Or $a(b - 1)(c - 1) - 1(1 - a)(c - 1) - 1(1 - a)(b - 1) = 0$

i.e., $a(1 - b)(1 - c) + (1 - a)(1 - c) + (1 - a)(1 - b) = 0$ **1**

Dividing both the sides by $(1 - a)(1 - b)(1 - c)$, we get

$$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

i.e.,
$$-1 + \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

i.e.,
$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$ **1**

[CBSE Marking Scheme 2016]

Commonly Made Error

- Some candidates do mistake while doing scalar triple product.
- The right product is $\left[\vec{a}\ \vec{b}\ \vec{c}\right] = \vec{a}\cdot\left[\vec{b}\times\vec{c}\right]$
- but student do mistake $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \left[\vec{a}\,.\vec{b}\right] \times \vec{c}$ or $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \left[\vec{a}\times\vec{c}\right]\times\vec{b}$

Q. 10. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then prove that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$, and hence show that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. R&U **[SQP 2017-18]**

Sol. $\vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{a}\times\vec{0}$ **1**

or $\vec{a}\times\vec{b}+\vec{a}\times\vec{c} = \vec{0}$ ½

or $\vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ½

$\vec{b}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{b}\times\vec{0}$ ½

or $\vec{b}\times\vec{c} = \vec{a}\times\vec{b}$ ½

$\therefore \quad \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ ½

$[\vec{a}\ \ \vec{b}\ \ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = 0$

[As the scalar triple product of three vectors is zero if any two of them are equal.] ½

[CBSE Marking Scheme 2017-18]

Q. 11. Find the value of λ, if four points with position vectors $3\hat{i}+6\hat{j}+9\hat{k}$, $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}+3\hat{j}+\hat{k}$ and $4\hat{i}+6\hat{j}+\lambda\hat{k}$ are coplanar. R&U **[O.D. Set I 2017]**

Sol. Given points, A, B, C, D are coplanar, if the vectors $\vec{AB}, \vec{AC}$ and $\vec{AD}$ are coplanar, *i.e.,*

$\vec{AB} = -2\hat{i} - 4\hat{j} - 6\hat{k}$, $\vec{AC} = -\hat{i} - 3\hat{j} - 8\hat{k}$, $\vec{AD} = \hat{i} + (\lambda - 9)\,\hat{k}$ are coplanar 1½

i.e.,
$$\begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda-9 \end{vmatrix} = 0$$

or $-2[-3\lambda + 27] + 4[-\lambda + 17] - 6(3) = 0$ **1+1**

or $\lambda = 2.$ **[CBSE Marking Scheme 2017]** ½

OR

$\overrightarrow{OA} = 3\hat{i} + 6\hat{j} + 9\hat{k}$

$\overrightarrow{OB} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{OC} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\overrightarrow{OD} = 4\hat{i} + 6\hat{j} + \lambda\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\hat{i} - 4\hat{j} - 6\hat{k} = \vec{a}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\hat{i} - 3\hat{j} - 8\hat{k} = \vec{b}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \hat{i} + 0\hat{j} + (\lambda - 9)\hat{k} = \vec{c}$

Scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ $\because$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar.

$$\begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda-9 \end{vmatrix} = 0$$

$(-2)(-3\lambda + 27) + 4(-\lambda + 9 + 8) - 6(3) = 0$

$6\lambda - 54 - 4\lambda + 68 - 18 = 0$

$2\lambda = 54 + 18 - 68$

$2\lambda = 72 - 68 = 4$

$\lambda = 2$

4

[Topper's Answer 2017]

Q. 12. Find the value of x such that the points $A(3, 2, 1)$, $B(4, x, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar. R&U **[O.D. 2017]**

Sol. Points A, B, C and D are coplanar, then the vectors $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ must be coplanar.

$\overrightarrow{AB} = \hat{i} + (x-2)\hat{j} + 4\hat{k}$; $\overrightarrow{AC} = \hat{i} - 3\hat{k}$,

$\overrightarrow{AD} = 3\hat{i} + 3\hat{j} - 2\hat{k}$ **1½**

i.e., $\begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$ **1**

or $1(9) - (x-2)(7) + 4(3) = 0$ or $x = 5$. **1½**

[CBSE Marking Scheme 2017]

Q. 13. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

(i) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar.

(ii) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. R&U **[Delhi 2017]**

Sol. $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_2 - c_3$ **1**

(i) $c_1 = 1, c_2 = 2$

$[\vec{a}\ \vec{b}\ \vec{c}] = 2 - c_3$ **1**

$\therefore$ $\vec{a}, \vec{b}, \vec{c}$ are coplanar $[\vec{a}\ \vec{b}\ \vec{c}] = 0$ or $c_3 = 2$ **1**

(ii) $c_2 = -1, c_3 = 1$

$[\vec{a}\ \vec{b}\ \vec{c}] = c_2 - c_3 = -2 \neq 0$

or No value of c_1 can make $\vec{a}, \vec{b}, \vec{c}$ coplanar **1**

[CBSE Marking Scheme 2017]

Q. 14. If four points A, B, C and D with position vectors $4\hat{i} + 3\hat{j} + 3\hat{k}$, $5\hat{i} + x\hat{j} + 7\hat{k}$, $5\hat{i} + 3\hat{j}$ and $7\hat{i} + 6\hat{j} + \hat{k}$ respectively are coplanar, then find the value of x. R&U **[Delhi Comptt. 2017]**

Sol. $\overrightarrow{AB} = \hat{i} + (x-3)\hat{j} + 4\hat{k}$

$\overrightarrow{AC} = \hat{i} - 3\hat{k}$

$\overrightarrow{AD} = 3\hat{i} + 3\hat{j} - 2\hat{k}$ **1½**

As A, B, C and D are coplanar

$\therefore$ $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0$

i.e., $\begin{vmatrix} 1 & x-3 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$ **1½**

$9 - (x-3)(7) + 12 = 0$

which gives

$x = 6$ **1**

[CBSE Marking Scheme 2017]

Long Answer Type Question-II (6 marks each)

Q. 1. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, then prove that :

(i) $\vec{a} = \pm 2(\vec{b} \times \vec{c})$

(ii) $[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}] = \pm 1$. Ⓐ **[S.Q.P. 2015-16]**

Sol. (i) As given

$$\vec{a}\cdot\vec{b} = 0,\ \vec{a}\cdot\vec{c} = 0 \;=\; \vec{a} \perp \text{both } \vec{b} \text{ and } \vec{c}$$

(as $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero vectors) **1**

or $\vec{a} \parallel (\vec{b} \times \vec{c})$

Let $\vec{a} = \lambda(\vec{b} \times \vec{c})$,

then $|\vec{a}| = |\lambda||(\vec{b}\times\vec{c})|$

or $\dfrac{|\vec{a}|}{|(\vec{b}\times\vec{c})|} = |\lambda|$

or $|\lambda| = \dfrac{1}{\sin\frac{\pi}{6}} = 2$

∴ $\lambda = \pm 2$

∴ $\vec{a} = \pm 2(\vec{b}\times\vec{c})$

Hence proved. 2

(ii) Now $[\vec{a}+\vec{b} \;\; \vec{b}+\vec{c} \;\; \vec{c}+\vec{a}]$

$$= [(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})]\cdot(\vec{c}+\vec{a})$$

$$= (\vec{a}\times\vec{b})\cdot\vec{c} + (\vec{b}\times\vec{c})\cdot\vec{a} \quad \mathbf{1}$$

(As the scalar triple product = 0, if any two vectors are equal)

Hence,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) + (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{c}\cdot(\vec{b}\times\vec{c})$$

$$= 2\vec{a}(\vec{b}\times\vec{c}) \quad \mathbf{1 + ½}$$

$$= 2\vec{a}\left(\pm\frac{1}{2}\vec{a}\right)$$

$$= \pm 1 \quad ½$$

Hence proved.

[CBSE Marking Scheme 2015]

Q. 2. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined an angle θ, then prove that $\tan\frac{\theta}{2} = \dfrac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$.

Sol. Given $|\vec{a}| = |\vec{b}| = 1$ & θ is angle between $\vec{a}$ and $\vec{b}$

$$\left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right) = \vec{a}.\vec{a} - \vec{b}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 - 2\vec{a}\vec{b} + |\vec{b}|^2$$

$$= 1 - 2\vec{a}.\vec{b} + 1 \quad \mathbf{1}$$

$$|\vec{a}-\vec{b}|^2 = 2 - 2\vec{a}.\vec{b}$$

$$= 2 - 2\,|a|\,|b|\,\cos\theta$$

$$|\vec{a}-\vec{b}|^2 = 2(1-\cos\theta) = 2\left(2\sin^2\frac{\theta}{2}\right) \quad \mathbf{1}$$

$$|\vec{a}-\vec{b}|^2 = 4\sec^2\frac{\theta}{2}$$

$$2\sin\frac{\theta}{2} = |\vec{a}-\vec{b}|$$

$$\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a}-\vec{b}| \quad \mathbf{1}$$

Now $\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right) = \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2.\vec{a}.\vec{b} + |\vec{b}|$$

$$|\vec{a}+\vec{b}|^2 = 1 + 2.\vec{a}.\vec{b} + 1 \quad ½$$

$$|\vec{a}+\vec{b}|^2 = 2 + 2|\vec{a}||\vec{b}|\cos\theta$$

$$|\vec{a}+\vec{b}|^2 = 2(1+\cos\theta)$$

$$|\vec{a}+\vec{b}|^2 = 4\cos^2\frac{\theta}{2} \quad ½$$

$$|\vec{a}+\vec{b}|^2 = 2\cos\frac{\theta}{2}$$

$$\cos\frac{\theta}{2} = \frac{1}{2}\left|\hat{a}+\hat{b}\right|$$

$$\tan\frac{\theta}{2} = \frac{\left|\vec{a}-\vec{b}\right|}{\left|\vec{a}+\vec{b}\right|}$$

Hence Proved. 2

Q. 3. If with reference to right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ and $\vec{\alpha} = 3\hat{i} - \hat{j}$,, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

Sol. Given $\vec{\beta_1}$ || to $\vec{\alpha}$

$\therefore$ $\vec{\beta_1} = \lambda\vec{\alpha}$

$\vec{\beta_1} = \lambda\left(3\hat{i} - \hat{j}\right)$ **2**

$\vec{\beta_2}$ is $\perp$ to $\vec{\alpha}$

$\therefore$ $\vec{\beta_2}.\vec{\alpha} = 0$

Given $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2} \Rightarrow \vec{\beta_2} = \left(2\hat{i} + \hat{j} - 3\hat{k}\right) - \lambda\left(3\hat{i} - \hat{j}\right)$ **2**

$\vec{\beta_2} = (2 - 3\lambda)\, i + (1 + \lambda)\ \hat{i} - 3\hat{k}$

$\vec{\beta_2}\cdot\alpha = [(2 - 3\lambda)\, i + (1 + \lambda) - 3k].[3i - j] = 0$

$3(2 - 3\lambda) - (1 + \lambda) = 0$

$5 - 10\lambda = 0$

$\lambda = \frac{1}{2}$

$\vec{\beta_1} = \frac{1}{2}\,(3i - j)\ \&\ \vec{\beta_2} = \frac{1}{2}\ \hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$ **2**

Commonly Made Errors

- Some students do dot product first then cross product which is wrong.
- The right method is to do the cross product first then dot product in scalar triple product.

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