Vector Algebra

(2025)

1.

If vector $\vec{a} = 3\hat{i} + 2\hat{j} - \hat{k}$ and vector $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, then which of the following is correct ?

(1 Marks) (2025)

(A) $\vec{a} \parallel \vec{b}$

(B) $\vec{a} \perp \vec{b}$

(C) $|\vec{b}| > |\vec{a}|$

(D) $|\vec{a}| = |\vec{b}|$

2.

$\int_{-1}^{1} \frac{|x|}{x} dx, x \neq 0$ is equal to

(1 Marks) (2025)

(A) -1

(B) 0

(C) 1

(D) 2

3.

If $\int \frac{2^{\frac{1}{x}}}{x^2} dx = k \cdot 2^{\frac{1}{x}} + C$, then k is equal to

(1 Marks) (2025)

(A) $\frac{-1}{\log 2}$

(B) $-\log 2$

(C) -1

(D) $\frac{1}{2}$

4.

The diagonals of a parallelogram are given by $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - \hat{k}$. Find the area of the parallelogram.

(2 Marks) (2025)

5.

Two friends while flying kites from different locations, find the strings of their kites crossing each other. The strings can be represented by vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$. Determine the angle formed between the kite strings. Assume there is no slack in the strings.

(2 Marks) (2025)

6.

Verify that lines given by $\vec{r} = (1 - \lambda)\hat{i} + (\lambda - 2)\hat{j} + (3 - 2\lambda)\hat{k}$ and $\vec{r} = (\mu + 1)\hat{i} + (2\mu - 1)\hat{j} - (2\mu + 1)\hat{k}$ are skew lines. Hence, find shortest distance between the lines.

(3 Marks) (2025)

Answers

1.

(B) $\vec{a} \perp \vec{b}$

2. (B) 0

3.

(A) $\frac{-1}{\log 2}$

4.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 1 & 3 & -1\end{vmatrix}=-2\hat{i}+3\hat{j}+7\hat{k}$$

$$\text{Area of parallelogram}=\frac{1}{2}\left|\vec{a}\times\vec{b}\right|$$

$$=\frac{1}{2}\sqrt{(-2)^2+3^2+7^2}=\frac{\sqrt{62}}{2}$$

5.

Let the required angle between the kite strings be θ.

$$\text{Then,}\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow\cos\theta==\frac{\left(3\hat{i}+\hat{j}+2\hat{k}\right)\left(2\hat{i}-2\hat{j}+4\hat{k}\right)}{\sqrt{9+1+4}\sqrt{4+4+16}}=\frac{12}{\sqrt{336}}=\frac{3}{\sqrt{21}}$$

$$\Rightarrow\theta=\cos^{-1}\left(\frac{12}{\sqrt{336}}\right)\text{ or }\cos^{-1}\left(\frac{3}{\sqrt{21}}\right)$$

6. Rewriting the lines, we get

$$\vec{r}=\left(\hat{i}-2\hat{j}+3\hat{k}\right)+\lambda\left(-\hat{i}+\hat{j}-2\hat{k}\right)\text{ and }\vec{r}=\left(\hat{i}-\hat{j}-\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}-2\hat{k}\right)$$

Let $\vec{a}_1=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{a}_2=\hat{i}-\hat{j}-\hat{k}$, $\vec{b}_1=-\hat{i}+\hat{j}-2\hat{k}$, $\vec{b}_2=\hat{i}+2\hat{j}-2\hat{k}$

Note that the dr'sof given lines are not proportional so, they are not parallel lines.

The lines will be skew if they do not intersect each other also.

Here $\vec{a}_2 - \vec{a}_1 = \hat{j} - 4\hat{k}$, $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = 2\hat{i} - 4\hat{j} - 3\hat{k}$

Consider $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)$

$= (\hat{j} - 4\hat{k}).(2\hat{i} - 4\hat{j} - 3\hat{k}) = 8 \neq 0$

Hence lines will not intersect. So, the lines are skew.

$$\text{Shortest Distance} = \frac{\left|(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)\right|}{\left|\vec{b}_1 \times \vec{b}_2\right|}$$

$$= \frac{8}{\sqrt{4+16+9}} = \frac{8}{\sqrt{29}}$$

(2024)

1.

Let θ be the angle between two unit vectors $\hat{a}$ and $\hat{b}$ such that $\sin\theta = \frac{3}{5}$.

Then, $\hat{a} \cdot \hat{b}$ is equal to :

(2024)

(A) $\pm\frac{3}{5}$ (B) $\pm\frac{3}{4}$

(C) $\pm\frac{4}{5}$ (D) $\pm\frac{4}{3}$

Ans.

(C) $\pm\frac{4}{5}$

2. The vector with terminal point A (2, – 3, 5) and initial point B (3, – 4, 7) is :

(2024)

(A) $\hat{i} - \hat{j} + 2\hat{k}$ (B) $\hat{i} + \hat{j} + 2\hat{k}$

(C) $-\hat{i} - \hat{j} - 2\hat{k}$ (D) $-\hat{i} + \hat{j} - 2\hat{k}$

Ans.

(D) $-\hat{i} + \hat{j} - 2\hat{k}$

3.

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}\,|\vec{a}|$.

(2024)

Ans.

$(\vec{a} + \vec{b}).\vec{a} = 0 \Rightarrow |\vec{a}|^2 + \vec{b}.\vec{a} = 0$ -----------------(1)

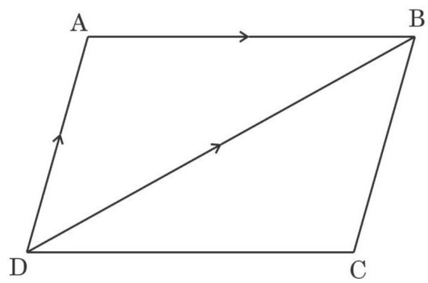
$(2\vec{a} + \vec{b}).\vec{b} = 0 \Rightarrow 2\vec{a}.\vec{b} + |\vec{b}|^2 = 0$ -------------(2)

$2(-|\vec{a}|^2) + |\vec{b}|^2 = 0$ {Using (1) and (2)}

$|\vec{b}|^2 = 2|\vec{a}|^2 \Rightarrow |\vec{b}| = \sqrt{2}|\vec{a}|$

4.

In the given figure, ABCD is a parallelogram. If $\overrightarrow{AB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\overrightarrow{DB} = 3\hat{i} - 6\hat{j} + 2\hat{k}$, then find $\overrightarrow{AD}$ and hence find the area of parallelogram ABCD.



(2024)

Ans.

$$\overrightarrow{AD} + \overrightarrow{DB} = \overrightarrow{AB}$$

$$\overrightarrow{AD} = (2\hat{i} - 4\hat{j} + 5\hat{k}) - (3\hat{i} - 6\hat{j} + 2\hat{k})$$

$$= -\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{AD} \times \overrightarrow{AB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 3 \\ 2 & -4 & 5 \end{vmatrix} = 22\hat{i} + 11\hat{j}$$

$$\text{Area} = |\overrightarrow{AD} \times \overrightarrow{AB}| = |22\hat{i} + 11\hat{j}|$$

$$= \sqrt{605} \text{ or } 11\sqrt{5}$$

Previous Years' CBSE Board Questions

10.2 Some Basic Concepts

VSA (1 mark)

1. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis. *(AI 2014)* Ev

SA I (2 marks)

2. Find a vector $\vec{r}$ equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units. *(2020)* An

10.3 Types of Vectors

MCQ

3. Two vectors $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ are collinear if
   (a) $a_1b_1+a_2b_2+a_3b_3=0$ (b) $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$
   (c) $a_1=b_1, a_2=b_2, a_3=b_3$
   (d) $a_1+a_2+a_3=b_1+b_2+b_3$ *(2023)*

4. The value of p for which $p(\hat{i}+\hat{j}+\hat{k})$ is a unit vector is
   (a) 0 (b) $\frac{1}{\sqrt{3}}$ (c) 1 (d) $\sqrt{3}$
   (2020) Ap

10.4 Addition of Vectors

MCQ

5. ABCD is a rhombus, whose diagonals intersect at E. Then $\overline{EA}+\overline{EB}+\overline{EC}+\overline{ED}$ equals
   (a) $\vec{0}$ (b) $\overline{AD}$ (c) $2\overline{BC}$ (d) $2\overline{AD}$
   (2020) Ap

10.5 Multiplication of a Vector by a Scalar

MCQ

6. A unit vector along the vector $4\hat{i}-3\hat{k}$ is
   (a) $\frac{1}{7}(4\hat{i}-3\hat{k})$ (b) $\frac{1}{5}(4\hat{i}-3\hat{k})$
   (c) $\frac{1}{\sqrt{7}}(4\hat{i}-3\hat{k})$ (d) $\frac{1}{\sqrt{5}}(4\hat{i}-3\hat{k})$ *(2023)*

VSA (1 mark)

7. The position vector of two points A and B are $\overline{OA}=2\hat{i}-\hat{j}-\hat{k}$ and $\overline{OB}=2\hat{i}-\hat{j}+2\hat{k}$, respectively. The position vector of a point P which divides the line segment joining A and B in the ratio 2 : 1 is ________. *(2020)* An

8. Find the position vector of a point which divides the join of points with position vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+\vec{b}$ externally in the ratio 2 : 1. *(Delhi 2016)* An

9. Write the position vector of the point which divides the join of points with position vectors $3\vec{a}-2\vec{b}$ and $2\vec{a}+3\vec{b}$ in the ratio 2 : 1. *(AI 2016)*

10. Find the unit vector in the direction of the sum of the vectors $2\hat{i}+3\hat{j}-\hat{k}$ and $4\hat{i}-3\hat{j}+2\hat{k}$. *(Foreign 2015)*

11. Find a vector in the direction of $\vec{a}=\hat{i}-2\hat{j}$ that has magnitude 7 units. *(Delhi 2015C)*

12. Write the direction ratios of the vector $3\vec{a}+2\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}-4\hat{j}+5\hat{k}$. *(AI 2015C)* An

13. Write a unit vector in the direction of the sum of the vectors $\vec{a}=2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-7\hat{k}$. *(Delhi 2014)*

14. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel. *(AI 2014)* An

15. Find a vector in the direction of vector $2\hat{i}-3\hat{j}+6\hat{k}$ which has magnitude 21 units. *(Foreign 2014)*

16. Write a unit vector in the direction of vector $\overline{PQ}$, where P and Q are the points(1, 3, 0) and (4, 5, 6) respectively. *(Foreign 2014)*

17. Write a vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 units. *(Delhi 2014C)* An

SA I (2 marks)

18. X and Y are two points with position vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio 2 : 1 externally. *(AI 2019)* Cr

LA I (4 marks)

19. The two vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through A. *(Delhi 2016, Foreign 2015)*

10.6 Product of Two Vectors

MCQ

20. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b}\geq 0$ only when
   (a) $0<\theta<\frac{\pi}{2}$ (b) $0\leq\theta\leq\frac{\pi}{2}$
   (c) $0<\theta<\pi$ (d) $0\leq\theta\leq\pi$ *(2023)*

21. The magnitude of the vector $6\hat{i}-2\hat{j}+3\hat{k}$ is
   (a) 1 (b) 5 (c) 7 (d) 12
   (2023)

22. If the projection of $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ on $\vec{b}=2\hat{i}+\lambda\hat{k}$ is zero, then the value of λ is
(a) 0 (b) 1
(c) $\frac{-2}{3}$ (d) $\frac{-3}{2}$ *(2020)* An

23. If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually perpendicular directions, then
(a) $\hat{i}\cdot\hat{j}=1$ (b) $\hat{i}\times\hat{j}=1$
(c) $\hat{i}\cdot\hat{k}=0$ (d) $\hat{i}\times\hat{k}=0$ *(2020)* Ap

VSA (1 mark)

24. Find the magnitude of vector $\vec{a}$ given by $\vec{a}=(\hat{i}+3\hat{j}-2\hat{k})\times(-\hat{i}+3\hat{k})$. *(2021C)*

25. If $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=5\hat{i}-3\hat{j}-4\hat{k}$ then find the ratio $\frac{\text{projection of vector } \vec{a} \text{ on vector } \vec{b}}{\text{projection of vector } \vec{b} \text{ on vector } \vec{a}}$. *(2020C)*

26. The area of the parallelogram whose diagonals are $2\hat{i}$ and $-3\hat{k}$ is _______ square units. *(2020)*

27. The value of λ for which the vectors $2\hat{i}-\lambda\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}-\hat{k}$ are orthogonal is _______. *(2020)*

28. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. *(2018)* An

29. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$. *(AI 2016)*

30. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then write the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$. *(NCERT, Foreign 2016)*

31. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=400$ and $|\vec{a}|=5$ then write the value of $|\vec{b}|$. *(Foreign 2016)*

32. If $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$. *(Delhi 2015)*

33. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors, then find the value of $|2\hat{a}+\hat{b}+\hat{c}|$. *(AI 2015)* Ev

34. Write a unit vector perpendicular to both the vectors $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}$. *(AI 2015)*

35. Find the area of a parallelogram whose adjacent sides are represented by the vectors $2\hat{i}-3\hat{k}$ and $4\hat{j}+2\hat{k}$. *(Foreign 2015)*

36. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ so that $\sqrt{2}\vec{a}-\vec{b}$ is a unit vector? *(Delhi 2015C)* An

37. Find the projection of the vector $\vec{a}=2\hat{i}+3\hat{j}+2\hat{k}$ on the vector $\vec{b}=2\hat{i}+2\hat{j}+\hat{k}$. *(AI 2015C)*

38. Find the projection of vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$. *(Delhi 2014)* U

39. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$ *(Delhi 2014)* Ap

40. If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}|=3, |\vec{b}|=\frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)* Cr

41. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}|=13$ and $|\vec{a}|=5$, find the value of $|\vec{b}|$. *(AI 2014)* An

42. Write the projection of the vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$. *(Foreign 2014)*

43. Write the value of $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$. *(Foreign 2014)*

44. Write the projection of the vector $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ on the vector $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$. *(Delhi 2014C)* Ap

45. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\vec{a}-\vec{b})$ is a unit vector. *(Delhi 2014C)*

46. Write the value of cosine of the angle which the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ makes with y-axis. *(Delhi 2014C)* Ap

47. If $|\vec{a}|=8, |\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$, find the angle between $\vec{a}$ and $\vec{b}$. *(AI 2014C)*

48. Find the angle between x-axis and the vector $\hat{i}+\hat{j}+\hat{k}$. *(AI 2014C)* Cr

SA I (2 marks)

49. If $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+\hat{k}$, then find a unit vector along the vector $\vec{a}\times\vec{b}$. *(2023)*

50. If the vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=3, |\vec{b}|=\frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. *(2023)*

51. Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$. *(2023)*

52. Write the projection of the vector $(\vec{b}+\vec{c})$ on the vector $\vec{a}$ where, $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. *(Term II, 2021-22)* Ap

53. If $\vec{a}=\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=-\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{c}=-\hat{i}+2\hat{j}-\hat{k}$ are three vectors, then find a vector perpendicular to both the vectors $(\vec{a}+\vec{b})$ and $(\vec{b}-\vec{c})$. *(Term II, 2021-22C)*

54. $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|2\vec{a}+3\vec{b}| = |3\vec{a}-2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$. *(Term II, 2021-22)*

55. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=400$ and $|\vec{b}|=5$, then find the value of $|\vec{a}|$. *(Term II, 2021-22)* Ap

56. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}, \vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$, then find $|\vec{b}|$. *(Term II, 2021-22)*

57. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$. *(Term II, 2021-22C)*

58. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}+\vec{b}|=|\vec{b}|$, then prove that $(\vec{a}+2\vec{b})$ is perpendicular to $\vec{a}$. *(Term II, 2021-22)*

59. If the sides AB and BC of a parallelogram $ABCD$ are represented as vectors $\overline{AB}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\overline{BC}=\hat{i}+2\hat{j}+3\hat{k}$, then find the unit vector along diagonal AC. *(2021C)*

60. Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a}=5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b}=7\hat{i}+6\hat{j}+2\hat{k}$. *(2020)*

61. Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two non-zero vectors $\vec{a}$ and $\vec{b}$. *(2020 C)*

62. Show that for any two non-zero vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$ iff $\vec{a}$ and $\vec{b}$ are perpendicular vectors. *(2020)*

63. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}, 3\hat{i}+7\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+2\hat{k}$ form the sides of a right-angled triangle. *(2020)*

64. If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. *(Delhi 2019)*

65. Let $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$ be two vectors. Show that the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other. *(AI 2019)*

66. Find a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ where $\vec{a}=4\hat{i}-\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{j}+\hat{k}$ *(2019 C)*

67. If $|\vec{a}|=2, |\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$. *(2019)* Ap

68. For any two vectors, $\vec{a}$ and $\vec{b}$, prove that $(\vec{a}\times\vec{b})^2=\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2$ *(2019)* An

69. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find $\sin\theta$. *(2018)*

SA II (3 marks)

70. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$, then find a unit vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$. *(2023)*

71. Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=2$. *(2023)*

72. The two adjacent sides of a parallelogram are represented by $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram also. *(Term II, 2021-22)*

73. If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitude, then prove that the vector $(2\vec{a}+\vec{b}+2\vec{c})$ is equally inclined to both $\vec{a}$ and $\vec{c}$. Also, find the angle between $\vec{a}$ and $(2\vec{a}+\vec{b}+2\vec{c})$. *(Term II, 2021-22)*

74. If $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=4$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then find the value of $(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$ *(Term II, 2021-22)*

75. If $\vec{a}$ and $\vec{b}$ are two vectors of equal magnitude and α is the angle between them, then prove that $\dfrac{|\vec{a}+\vec{b}|}{|\vec{a}-\vec{b}|}=\cot\left(\dfrac{\alpha}{2}\right)$. *(Term II, 2021-22)*

LA I (4 marks)

76. If $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram. *(2020)*

77. Using vectors, find the area of the triangle ABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$ *(NCERT Exemplar, 2020)*

78. Prove that three points A, B and C with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively are collinear if and only if $(\vec{b}\times\vec{c})+(\vec{c}\times\vec{a})+(\vec{a}\times\vec{b})=\vec{0}$. *(2020 C)*

79. The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$ *(2019, AI 2014)*

80. If $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overline{AB}$ and $\overline{CD}$ are collinear or not. *(Delhi 2019)*

81. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}|=1, |\vec{b}|=2, |\vec{c}|=3$. If the projection of $\vec{b}$ along $\vec{a}$ is equal to the projection of $\vec{c}$ along $\vec{a}$; and $\vec{b}, \vec{c}$ are perpendicular to each other, then find $|3\vec{a}-2\vec{b}+2\vec{c}|$. *(2019)*

82. Let $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}, \vec{b}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a}=21$. *(2018)*

83. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a}+\vec{b}+\vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. *(Delhi 2017)*

84. Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle. *(AI 2017)*

85. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. *(AI 2016)*

86. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$. *(Foreign 2016)*

87. If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, find $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy$. *(Delhi 2015)*

88. If $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$, then find a unit vector perpendicular to both of the vectors $(\vec{a}-\vec{b})$ and $(\vec{c}-\vec{b})$. *(AI 2015)*

89. Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$. *(Delhi 2014)*

90. Find a unit vector perpendicular to both of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$. *(Foreign 2014)*

91. If $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{k}$, $\vec{c}=2\hat{j}-\hat{k}$ are three vectors, find the area of the parallelogram having diagonals $(\vec{a}+\vec{b})$ and $(\vec{b}+\vec{c})$. *(Delhi 2014C)*

92. Find the vector $\vec{p}$ which is perpendicular to both $\vec{\alpha}=4\hat{i}+5\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{p}\cdot\vec{q}=21$, where $\vec{q}=3\hat{i}+\hat{j}-\hat{k}$. *(AI 2014C)*

CBSE Sample Questions

10.5 Multiplication of a Vector by a Scalar

VSA (1 mark)

1. Find a unit vector in the direction opposite to $-\frac{3}{4}\hat{j}$. *(2020-21)*

2. Vector of magnitude 5 units and in the direction opposite to $2\hat{i}+3\hat{j}-6\hat{k}$ is _____. *(2020-21)* U

10.6 Product of Two Vectors

MCQ

3. The scalar projection of the vector $3\hat{i}-\hat{j}-2\hat{k}$ on the vector $\hat{i}+2\hat{j}-3\hat{k}$ is

   (a) $\frac{7}{\sqrt{14}}$ (b) $\frac{7}{14}$ (c) $\frac{6}{13}$ (d) $\frac{7}{2}$

   (2022-23)

4. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2, |\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=4$, then $|\vec{a}-2\vec{b}|$ is equal to

   (a) $\sqrt{2}$ (b) $2\sqrt{6}$ (c) 24 (d) $2\sqrt{2}$

   (2022-23)

VSA (1 mark)

5. Find the area of the triangle whose two sides are represented by the vectors $2\hat{i}$ and $-3\hat{j}$. *(2020-21)* Ap

6. Find the angle between the unit vectors $\hat{a}$ and $\hat{b}$, given that $|\hat{a}+\hat{b}|=1$. *(2020-21)*

SA I (2 marks)

7. Find $|\vec{x}|$, if $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$, where $\vec{a}$ is a unit vector. *(2022-23)* An

8. If $\vec{a}$ and $\vec{b}$ are unit vectors, then prove that $|\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$, where θ is the angle between them. *(Term II, 2021-22)* Ev

9. Find the area of the parallelogram whose one side and a diagonal are represented by coinitial vectors $\hat{i}-\hat{j}+\hat{k}$ and $4\hat{i}+5\hat{k}$ respectively. *(2020-21)*

SA II (3 marks)

10. If $\vec{a}\neq\vec{0}, \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}, \vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, then show that $\vec{b}=\vec{c}$. *(Term II, 2021-22)* An

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. Here, $l=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, $m=\cos\frac{\pi}{2}=0$, $n=\cos\theta$

Since, $l^2+m^2+n^2=1$

$\Rightarrow \frac{1}{2}+0+\cos^2\theta=1 \Rightarrow \cos^2\theta=1-\frac{1}{2}=\frac{1}{2}$

$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}$

Commonly Made Mistake

- $\cos\theta = \pm\frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4}, \frac{3\pi}{4} \quad \therefore \theta = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ The vector of magnitude $5\sqrt{2}$ is

$\vec{a} = 5\sqrt{2}(l\hat{i} + m\hat{j} + n\hat{k})$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right) = 5(\hat{i} + \hat{k})$ [$\because \theta$ is an acute angle]

2. We have, $|\vec{r}| = 3\sqrt{3}$

Since, $\vec{r}$ is equally inclined to three axes, so direction cosine of unit vector $\vec{r}$ will be same. *i.e.*, $l = m = n$

As we know that $l^2 + m^2 + n^2 = 1$ ...(i)

$l^2 + l^2 + l^2 = 1 \Rightarrow 3l^2 = 1$

$\Rightarrow l = \pm\frac{1}{\sqrt{3}} = m = n$ ...(ii)

We have $\overline{OP} = \pm\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)$ $\left\{\because \hat{p} = \frac{\vec{r}}{|\vec{r}|}\right\}$

$\vec{r} = |\vec{r}|\overline{OP}$ $\{\because |\vec{r}| = 3\sqrt{3}$ (given)$\}$

$= \pm 3\sqrt{3} \times \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) \Rightarrow \vec{r} = \pm 3(\hat{i} + \hat{j} + \hat{k})$

Answer Tips

- If a vector is equally inclined to axes, then its direction cosines are equal.

3. (b)

4. (b) : Let $\vec{a} = (\hat{i} + \hat{j} + \hat{k})$

So, unit vector of $\vec{a} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$

$\therefore$ The value of p is $\frac{1}{\sqrt{3}}$.

5. (a) : $\overline{EA} + \overline{EB} + \overline{EC} + \overline{ED} = \overline{EA} + \overline{EB} - \overline{EA} - \overline{EB}$

[As diagonals of a rhombus bisect each other]

$= \vec{0}$

6. (b): Let $\vec{v} = 4\hat{i} - 3\hat{k}$

$\therefore |\vec{v}| = \sqrt{4^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

Now, $\hat{v}$ = unit vector along $\vec{v}$

$= \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(4\hat{i} - 3\hat{k})$

7. Required position vector of point P

$= \frac{1(2\hat{i} - \hat{j} - \hat{k}) + 2(2\hat{i} - \hat{j} + 2\hat{k})}{2+1} = \frac{2\hat{i} - \hat{j} - \hat{k} + 4\hat{i} - 2\hat{j} + 4\hat{k}}{3}$

$= \frac{1}{3}(6\hat{i} - 3\hat{j} + 3\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$

Concept Applied

- If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B respectively, then the position vector of $R(\vec{r})$ which divides $\overline{AB}$ internally in the ratio $m : n$ is $\frac{m\vec{b} + n\vec{a}}{m+n}$

8. Required position vector

$= \frac{2\cdot(2\vec{a} + \vec{b}) - 1(\vec{a} - 2\vec{b})}{2-1} = \frac{4\vec{a} + 2\vec{b} - \vec{a} + 2\vec{b}}{1} = 3\vec{a} + 4\vec{b}$

9. Required position vector

$= \frac{2(2\vec{a} + 3\vec{b}) + 1(3\vec{a} - 2\vec{b})}{2+1} = \frac{7\vec{a} + 4\vec{b}}{3} = \frac{7}{3}\vec{a} + \frac{4}{3}\vec{b}$

10. Let $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 4\hat{i} - 3\hat{j} + 2\hat{k}$.

Then, the sum of the given vectors is

$\vec{c} = \vec{a} + \vec{b} = (2+4)\hat{i} + (3-3)\hat{j} + (-1+2)\hat{k} = 6\hat{i} + \hat{k}$

and $|\vec{c}| = |\vec{a} + \vec{b}| = \sqrt{6^2 + 1^2} = \sqrt{36+1} = \sqrt{37}$

$\therefore$ Unit vector, $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{6\hat{i} + \hat{k}}{\sqrt{37}} = \frac{6}{\sqrt{37}}\hat{i} + \frac{1}{\sqrt{37}}\hat{k}$

11. A unit vector in the direction of $\vec{a} = \hat{i} - 2\hat{j}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$= \frac{\hat{i} - 2\hat{j}}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$

$\therefore$ The required vector of magnitude 7 in the direction of $\vec{a} = 7\cdot\hat{a} = \frac{7}{\sqrt{5}}(\hat{i} - 2\hat{j})$.

12. $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$; $\vec{b} = 2\hat{i} - 4\hat{j} + 5\hat{k}$

$\therefore 3\vec{a} + 2\vec{b} = 3(\hat{i} + \hat{j} - 2\hat{k}) + 2(2\hat{i} - 4\hat{j} + 5\hat{k})$

$= (3\hat{i} + 3\hat{j} - 6\hat{k}) + (4\hat{i} - 8\hat{j} + 10\hat{k}) = 7\hat{i} - 5\hat{j} + 4\hat{k}$

$\therefore$ The direction ratios of the vector $3\vec{a} + 2\vec{b}$ are $7, -5, 4$.

13. We have, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$

Sum of the given vectors is

$\vec{c} = \vec{a} + \vec{b} = (2+2)\hat{i} + (2+1)\hat{j} + (-5-7)\hat{k} = 4\hat{i} + 3\hat{j} - 12\hat{k}$

and $|\vec{c}| = \sqrt{(4)^2 + (3)^2 + (-12)^2} = \sqrt{169} = 13$

$\therefore$ Unit vector, $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{4\hat{i} + 3\hat{j} - 12\hat{k}}{13} = \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$

14. Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} - 2p\hat{j} + 3\hat{k}$

For $\vec{a}$ and $\vec{b}$ to be parallel, $\vec{b} = \lambda\vec{a}$.

$\Rightarrow \hat{i} - 2p\hat{j} + 3\hat{k} = \lambda(3\hat{i} + 2\hat{j} + 9\hat{k}) = 3\lambda\hat{i} + 2\lambda\hat{j} + 9\lambda\hat{k}$

$\Rightarrow 1 = 3\lambda, -2p = 2\lambda, 3 = 9\lambda \Rightarrow \lambda = \frac{1}{3}$ and $p = -\lambda = -\frac{1}{3}$

Concept Applied

- Two vectors $\vec{a}$ and $\vec{b}$ are parallel iff one of them is scalar multiple of other.

15. Let $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

A vector in the direction of $\vec{a}$ with a magnitude of $21 = 21\,\hat{a}$

$\therefore$ Required vector $= 21 \times \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{2^2 + (-3)^2 + 6^2}}$

$= 21 \times \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7} = 6\hat{i} - 9\hat{j} + 18\hat{k}$

16. We have, $\overline{PQ} = \overline{OQ} - \overline{OP}$

$= (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 3\hat{j}) = 3\hat{i} + 2\hat{j} + 6\hat{k}$

Required unit vector $= \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{3\hat{i} + 2\hat{j} + 6\hat{k}}{7}$

Concept Applied

- Unit vector of $\overline{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|}$

17. Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

The vector in the direction of $\vec{a}$ with magnitude 9 units $= 9\hat{a}$

$\therefore$ Required vector $= 9 \times \frac{\vec{a}}{|\vec{a}|} = 9 \times \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}}$

$= \frac{9}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) = 3\hat{i} - 6\hat{j} + 6\hat{k}$

Answer Tips

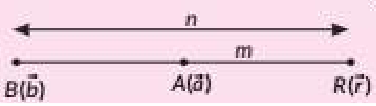
- The vector in direction of $\vec{a} = |\vec{a}| \cdot \hat{a}$

18. Position vector which divides the line segment joining points with position vectors $3\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ in the ratio 2 : 1 externally is given by

$\frac{2(\vec{a} - 3\vec{b}) - 1(3\vec{a} + \vec{b})}{2-1} = \frac{2\vec{a} - 6\vec{b} - 3\vec{a} - \vec{b}}{1} = -\vec{a} - 7\vec{b}$

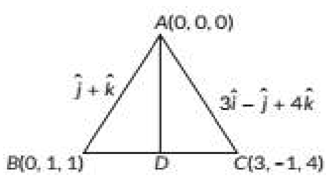
Concept Applied

- If $\vec{a}$ and $\vec{b}$ are position vectors of two points A and B respectively, then the position vector of $R(\vec{r})$ which divides $\overline{AB}$ externally in the ratio $m : n$ is $\frac{m\vec{b} - n\vec{a}}{m-n}$.



19. Take A to be as origin (0, 0, 0).

$\therefore$ Coordinates of B are (0, 1, 1) and coordinates of C are (3, –1, 4).



Let D be the mid point of BC and AD is a median of ΔABC.

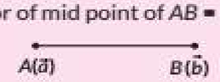
$\therefore$ Coordinates of D are $\left(\frac{3}{2}, 0, \frac{5}{2}\right)$

So, length of $AD = \sqrt{\left(\frac{3}{2} - 0\right)^2 + (0)^2 + \left(\frac{5}{2} - 0\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{25}{4}} = \frac{\sqrt{34}}{2}$ units

Concept Applied

- Position vector of mid point of $AB = \frac{\vec{a} + \vec{b}}{2}$



20. (b): Given, $\vec{a} \cdot \vec{b} \geq 0$

$\Rightarrow |\vec{a}||\vec{b}|\cos\theta \geq 0$

Assuming $|\vec{a}| \neq 0$ and $|\vec{b}| \neq 0$

$\Rightarrow \cos\theta \geq 0 \quad [\because |\vec{a}| \geq 0, |\vec{b}| \geq 0] \Rightarrow \theta \in \left[0, \frac{\pi}{2}\right]$

21. (c): Given vector is $6\hat{i} - 2\hat{j} + 3\hat{k}$

$\therefore$ Its magnitude $= \sqrt{6^2 + (-2)^2 + 3^2}$

$= \sqrt{36 + 4 + 9} = \sqrt{49} = 7$ units

22. (c): Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} + \lambda\hat{k}$

Since, projection of $\vec{a}$ on $\vec{b} = 0$

$\Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 0 \Rightarrow \frac{(\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (2\hat{i} + \lambda\hat{k})}{\sqrt{2^2 + \lambda^2}} = 0$

$\Rightarrow \frac{2 + 3\lambda}{\sqrt{4 + \lambda^2}} = 0 \Rightarrow 2 + 3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}$

23. (c): Since, $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular to each other.

$\therefore \hat{i} \cdot \hat{k} = 0$

24. Let $\vec{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 0\hat{j} + 3\hat{k})$

$\vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = (9 - 0)\hat{i} - (3 - 2)\hat{j} + (0 + 3)\hat{k} = 9\hat{i} - \hat{j} + 3\hat{k}$

$\therefore \quad |\vec{a}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{91}$

Answer Tips

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

25. Here $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 5\hat{i} - 3\hat{j} - 4\hat{k}$

Since projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

and projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

$\therefore \quad \vec{a} \cdot \vec{b} = (2\hat{i} - \hat{j} + 2\hat{k}) \cdot (5\hat{i} - 3\hat{j} - 4\hat{k})$

$\vec{a} \cdot \vec{b} = 10 + 3 - 8 = 13 - 8 = 5$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$

$|\vec{b}| = \sqrt{5^2 + (-3)^2 + (-4)^2} = \sqrt{25 + 9 + 16} = \sqrt{50} = 5\sqrt{2}$

Required ratio $= \frac{5/5\sqrt{2}}{5/3} = \frac{5}{5\sqrt{2}} \times \frac{3}{5} = \frac{3\sqrt{2}}{10}$

26. Given, two diagonals $\vec{d}_1$ and $\vec{d}_2$ are $2\hat{i}$ and $-3\hat{k}$ respectively.

$\therefore \quad \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 0 & -3 \end{vmatrix} = \hat{i}(0) - \hat{j}(-6 - 0) + \hat{k}(0) = 6\hat{j}$

So, area of the parallelogram = $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$

$= \frac{1}{2} \times 6 = 3$ sq. units

27. Let $\vec{a} = 2\hat{i} - \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$

We know, $\vec{a}$ and $\vec{b}$ are orthogonal iff $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow (2\hat{i} - \lambda\hat{j} + \hat{k})\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 0$

$\Rightarrow 2 - 2\lambda - 1 = 0 \Rightarrow 1 - 2\lambda = 0 \Rightarrow \lambda = \frac{1}{2}$

28. Given, $|\vec{a}| = |\vec{b}|, \theta = 60°$ and $\vec{a}\cdot\vec{b} = \frac{9}{2}$

Now, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \cos 60° = \frac{9/2}{|\vec{a}|^2} \Rightarrow \frac{1}{2} = \frac{9/2}{|\vec{a}|^2}$

$\Rightarrow |\vec{a}|^2 = 9 \Rightarrow |\vec{a}| = 3 \therefore |\vec{a}| = |\vec{b}| = 3$

Answer Tips

- $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cos\theta$

29. Given, $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$

Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are $\pm\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)$

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{i} - 2\hat{j} + 2\hat{k}$

$\therefore$ Unit vectors perpendicular to $\vec{a}$ and $\vec{b}$ are

$$\pm\frac{(-\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{(-1)^2 + (-2)^2 + (2)^2}} = \pm\left(-\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

So, there are two unit vectors perpendicular to the given vectors.

30. We have $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors.

Therefore, $|\vec{a}| = 1, |\vec{b}| = 1$ and $|\vec{c}| = 1$

Also, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (given)

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = 0 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow 1 + 1 + 1 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow 3 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0 \Rightarrow (\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = -\frac{3}{2}$

31. $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 400 \Rightarrow \{|\vec{a}||\vec{b}|\sin\theta\}^2 + \{|\vec{a}||\vec{b}|\cos\theta\}^2 = 400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2\sin^2\theta + |\vec{a}|^2|\vec{b}|^2\cos^2\theta = 400$

$\Rightarrow |\vec{a}|^2|\vec{b}|^2 = 400 \Rightarrow 25 \times |\vec{b}|^2 = 400$ [$\because |\vec{a}| = 5$]

$\Rightarrow |\vec{b}|^2 = 16 \Rightarrow |\vec{b}| = 4$

32. Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \frac{(7\hat{i} + \hat{j} - 4\hat{k})\cdot(2\hat{i} + 6\hat{j} + 3\hat{k})}{\sqrt{(2)^2 + (6)^2 + (3)^2}} = \frac{14 + 6 - 12}{7} = \frac{8}{7}$

33. Here $\hat{a}, \hat{b}$ and $\hat{c}$ are mutually perpendicular unit vectors.

$\Rightarrow |\hat{a}| = |\hat{b}| = |\hat{c}| = 1$ and $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = 0$ ...(1)

$\therefore |2\hat{a} + \hat{b} + \hat{c}|^2 = (2\hat{a} + \hat{b} + \hat{c}).(2\hat{a} + \hat{b} + \hat{c})$

$= 4\hat{a}\cdot\hat{a} + 2\hat{a}\cdot\hat{b} + 2\hat{a}\cdot\hat{c} + 2\hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b} + \hat{b}\cdot\hat{c} + 2\hat{c}\cdot\hat{a} + \hat{c}\cdot\hat{b} + \hat{c}\cdot\hat{c}$

$= 4|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 4\hat{a}\cdot\hat{b} + 2\hat{b}\cdot\hat{c} + 4\hat{a}\cdot\hat{c}$

($\because \hat{b}\cdot\hat{a} = \hat{a}\cdot\hat{b}, \hat{c}\cdot\hat{a} = \hat{a}\cdot\hat{c}, \hat{c}\cdot\hat{b} = \hat{b}\cdot\hat{c}$)

$= 4\cdot1^2 + 1^2 + 1^2$ [Using (1)]

$= 6$

$\therefore |2\hat{a} + \hat{b} + \hat{c}| = \sqrt{6}$.

34. Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{i} + \hat{j} + 0\hat{k} = -\hat{i} + \hat{j}$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$= \pm\frac{-\hat{i} + \hat{j}}{\sqrt{(-1)^2 + 1^2}} = \pm\frac{1}{\sqrt{2}}(-\hat{i} + \hat{j})$.

Key Points

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

35. Let $\vec{a} = 2\hat{i} - 3\hat{k}$ and $\vec{b} = 4\hat{j} + 2\hat{k}$

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a}\times\vec{b}|$.

Now, $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 12\hat{i} - 4\hat{j} + 8\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{(12)^2 + (-4)^2 + (8)^2} = \sqrt{144 + 16 + 64}$

$= \sqrt{224} = 4\sqrt{14}$ sq. units.

36. Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$.

$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \vec{a}\cdot\vec{b}$ ($\because |\vec{a}| = 1 = |\vec{b}|$) ...(i)

Now, $1 = |\sqrt{2}\vec{a} - \vec{b}|$

$\Rightarrow 1 = |\sqrt{2}\vec{a} - \vec{b}|^2 = (\sqrt{2}\vec{a} - \vec{b})\cdot(\sqrt{2}\vec{a} - \vec{b})$

$= 2|\vec{a}|^2 - \sqrt{2}\vec{a}\cdot\vec{b} - \vec{b}\cdot\sqrt{2}\vec{a} + |\vec{b}|^2$

$= 2 - 2\sqrt{2}\vec{a}\cdot\vec{b} + 1$ ($\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$)

$= 3 - 2\sqrt{2}\vec{a}\cdot\vec{b}$

$\Rightarrow \vec{a}\cdot\vec{b} = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$ [By using (i)]

$\therefore \theta = \pi/4$

37. Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(2\hat{i}+3\hat{j}+2\hat{k})\cdot(2\hat{i}+2\hat{j}+\hat{k})}{\sqrt{(2)^2+(2)^2+(1)^2}} = \frac{4+6+2}{\sqrt{9}} = \frac{12}{3} = 4$$

38. Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}, \vec{b} = 2\hat{i}-3\hat{j}+6\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ = $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(\hat{i}+3\hat{j}+7\hat{k})\cdot(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{(2)^2+(-3)^2+(6)^2}} = \frac{2-9+42}{\sqrt{49}} = \frac{35}{7} = 5$$

39. Given $|\vec{a}|=1=|\vec{b}|, |\vec{a}+\vec{b}|=1$

$\Rightarrow |\vec{a}+\vec{b}|^2=1 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=1 \Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=1$

$\Rightarrow |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=1 \Rightarrow 1+2\vec{a}\cdot\vec{b}+1=1$

$\Rightarrow 2\vec{a}\cdot\vec{b}=-1 \Rightarrow 2|\vec{a}|\cdot|\vec{b}|\cos\theta=-1 \Rightarrow 2\cdot1\cdot1\cos\theta=-1$

$\Rightarrow \cos\theta=-\frac{1}{2} \Rightarrow \theta=120°$

40. Given, $|\vec{a}|=3, |\vec{b}|=\frac{2}{3}$ and $|\vec{a}\times\vec{b}|=1$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta=1 \Rightarrow 3.\frac{2}{3}\sin\theta=1 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$

41. Given: $\vec{a}\perp\vec{b} \Rightarrow \vec{a}\cdot\vec{b}=0$

Also, $|\vec{a}|=5$ and $|\vec{a}+\vec{b}|=13$

$\Rightarrow |\vec{a}+\vec{b}|^2=13^2 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=169$

$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=169 \Rightarrow |\vec{a}|^2+0+0+|\vec{b}|^2=169$

$\Rightarrow |\vec{b}|^2=169-|\vec{a}|^2=169-5^2=144 \Rightarrow |\vec{b}|=12$

42. The projection of the vector $(\hat{i}+\hat{j}+\hat{k})$ along the vector $\hat{j}$ is $(\hat{i}+\hat{j}+\hat{k})\cdot\left(\frac{\hat{j}}{\sqrt{0^2+1^2+0^2}}\right)=1$

43. We have, $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$

$= \hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{k}+\hat{j}\times\hat{i}+\hat{k}\times\hat{i}+\hat{k}\times\hat{j}$

$= \hat{k}-\hat{j}+\hat{i}-\hat{k}+\hat{j}-\hat{i}=\vec{0}$.

44. Projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$$= \frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{(1)^2+(2)^2+(2)^2}} = \frac{2-2+2}{\sqrt{9}} = \frac{2}{3}$$

45. Let θ be the angle between the unit vectors $\vec{a}$ and $\vec{b}$

$\therefore \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\vec{a}\cdot\vec{b}$ ...(i) $(\because |\vec{a}|=|\vec{b}|=1)$

Now, $|\sqrt{3}\vec{a}-\vec{b}|=1 \Rightarrow |\sqrt{3}\vec{a}-\vec{b}|^2=1$

$\Rightarrow 3|\vec{a}|^2+|\vec{b}|^2-2\sqrt{3}\,\vec{a}\cdot\vec{b}=1 \Rightarrow 3+1-2\sqrt{3}|\vec{a}||\vec{b}|\cos\theta=1$

$\Rightarrow 3=2\sqrt{3}\cos\theta \Rightarrow \cos\theta=\frac{\sqrt{3}}{2} \Rightarrow \theta=\frac{\pi}{6}$

46. Let θ be the angle between the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and y-axis *i.e.*, $\vec{b}=\hat{j}$

$\therefore \cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{(\hat{i}+\hat{j}+\hat{k}).\hat{j}}{|\hat{i}+\hat{j}+\hat{k}||\hat{j}|} = \frac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2}} = \frac{1}{\sqrt{3}}$$

47. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

Given: $|\vec{a}|=8, |\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$

$\Rightarrow |\vec{a}||\vec{b}|\sin\theta=12 \Rightarrow 8\times3\times\sin\theta=12$

$\Rightarrow \sin\theta=\frac{12}{24}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{6}$.

48. Here, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and vector along x-axis is $\hat{i}$

$\therefore$ Angle between $\vec{a}$ and $\hat{i}$ is given by

$$\cos\theta = \frac{\vec{a}.\hat{i}}{\sqrt{1^2+1^2+1^2}.\sqrt{1^2}} = \frac{(\hat{i}+\hat{j}+\hat{k})\cdot\hat{i}}{\sqrt{3}\cdot1} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

49. We have, $\vec{a}=4\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+\hat{k}$

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 1\\ 2 & -2 & 1\end{vmatrix}$$

$= \hat{i}(-1+2)-\hat{j}(4-2)+\hat{k}(-8+2) = \hat{i}-2\hat{j}-6\hat{k}$

$|\vec{a}\times\vec{b}|=\sqrt{1^2+(-2)^2+(-6)^2}=\sqrt{41}$

Unit vector along $\vec{a}\times\vec{b}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$$=\frac{\hat{i}-2\hat{j}-6\hat{k}}{\sqrt{41}}=\frac{1}{\sqrt{41}}\hat{i}-\frac{2}{\sqrt{41}}\hat{j}-\frac{6}{\sqrt{41}}\hat{k}$$

50. We know that, $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

$\Rightarrow 1=3\times\frac{2}{3}\sin\theta$ $(\because \vec{a}\times\vec{b}$ is a unit vector$)$

$\Rightarrow \sin\theta=\frac{1}{2}=\sin30° \Rightarrow \theta=30°$

So, angle between $\vec{a}$ and $\vec{b}$ is 30°.

51. Area of parallelogram $=\vec{a}\times\vec{b} = \left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{matrix}\right\|$

$=|(-1+21)\hat{i}-(1-6)\hat{j}+(-7+2)\hat{k}|=|20\hat{i}+5\hat{j}-5\hat{k}|$

$=\sqrt{(20)^2+(5)^2+(-5)^2}$

$=\sqrt{400+25+25}=\sqrt{450}=15\sqrt{2}$ sq.units

52.

$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

$\vec{b} + \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

projection of $(\vec{b}+\vec{c})$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot(\vec{b}+\vec{c})}{|\vec{a}|}$

$= \dfrac{(2\hat{i} - 2\hat{j} + \hat{k})\cdot(3\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{2^2 + (-2)^2 + 1^2}}$

$= \dfrac{(2\times 3) + (-2\times 1) + (1\times 2)}{\sqrt{9}}$

$= \dfrac{6 - 2 + 2}{\sqrt{9}}$

$= \dfrac{6}{3}$

$= 2$ units

Answer: Projection of $(\vec{b}+\vec{c})$ on $\vec{a}$ = 2 units.

[Topper's Answer, 2022]

53. Here $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{j} - \hat{k}$

$\therefore\ \vec{a} + \vec{b} = (\hat{i} + \hat{j} - 2\hat{k}) + (-\hat{i} + 2\hat{j} + 2\hat{k}) = 3\hat{j}$

Now $(\vec{b} - \vec{c}) = (-\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 2\hat{j} - \hat{k}) = 3\hat{k}$

Vector perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{b}-\vec{c})$ is

$$(\vec{a}+\vec{b}) \times (\vec{b}-\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{vmatrix} = \hat{i}(9-0) - \hat{j}(0-0) + \hat{k}(0-0) = 9\hat{i}$$

$\therefore$ Unit vector perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{b}-\vec{c})$

$$= \frac{9\hat{i}}{\sqrt{9^2 + 0^2 + 0^2}} = \frac{9\hat{i}}{9} = \hat{i} + 0\hat{j} + 0\hat{k}$$

54. Given $\vec{a}$ and $\vec{b}$ are unit vectors

$\therefore\ |\vec{a}| = |\vec{b}| = 1$ ...(i)

Let θ be the angle between $\vec{a}$ and $\vec{b}$.

Also, $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$ (Given)

$\Rightarrow\ |2\vec{a} + 3\vec{b}|^2 = |3\vec{a} - 2\vec{b}|^2$

$\Rightarrow\ (2\vec{a} + 3\vec{b})\cdot(2\vec{a} + 3\vec{b}) = (3\vec{a} - 2\vec{b})\cdot(3\vec{a} - 2\vec{b})$

$\Rightarrow\ 4(\vec{a}\cdot\vec{a}) + 6(\vec{a}\cdot\vec{b}) + 6(\vec{b}\cdot\vec{a}) + 9(\vec{b}\cdot\vec{b})$

$= 9(\vec{a}\cdot\vec{a}) - 6(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{a}) + 4(\vec{b}\cdot\vec{b})$

$\Rightarrow\ 4|\vec{a}|^2 + 12(\vec{a}\cdot\vec{b}) + 9|\vec{b}|^2 = 9|\vec{a}|^2 - 12(\vec{a}\cdot\vec{b}) + 4|\vec{b}|^2$

$\Rightarrow\ 5|\vec{a}|^2 - 5|\vec{b}|^2 - 24|\vec{a}||\vec{b}|\cos\theta = 0$

$\Rightarrow\ 5\cdot 1 - 5\cdot 1 - 24\cos\theta = 0$ $(\because |\vec{a}| = |\vec{b}| = 1)$

$\Rightarrow\ \cos\theta = 0 \ \Rightarrow\ \theta = \dfrac{\pi}{2}$

Answer Tips

- If $\vec{a}$ is a unit vector, then $|\vec{a}| = 1$

55. We have, $|\vec{a}\times\vec{b}|^2 + |\vec{a}\cdot\vec{b}|^2 = 400$

$\Rightarrow\ |\vec{a}|^2|\vec{b}|^2\sin^2\theta + |\vec{a}|^2|\vec{b}|^2\cos^2\theta = 400$

$\Rightarrow\ |\vec{a}|^2|\vec{b}|^2 = 400 \quad (\because \sin^2\theta + \cos^2\theta = 1)$

$\Rightarrow\ |\vec{a}|^2 \times 25 = 400 \ \Rightarrow\ |\vec{a}|^2 = \dfrac{400}{25} = 16 \ \Rightarrow\ |\vec{a}| = 4$

56. Given : $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{a}\cdot\vec{b} = 1$ and $\vec{a}\times\vec{b} = \hat{j} - \hat{k}$

To find $|\vec{b}|$.

Let $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k}$

Since, $\vec{a}\cdot\vec{b} = 1$

$\Rightarrow\ (\hat{i} + \hat{j} + \hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k}) = 1 \Rightarrow x + y + z = 1$

and $\vec{a}\times\vec{b} = \hat{j} - \hat{k}$

$\Rightarrow\ (\hat{i} + \hat{j} + \hat{k})\times(x\hat{i} + y\hat{j} + z\hat{k}) = \hat{j} - \hat{k}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k} \Rightarrow \hat{i}(z-y) - \hat{j}(z-x) + \hat{k}(y-x) = \hat{j} - \hat{k}$$

$\Rightarrow\ x - z = 1, y - x = -1, z - y = 0$

$\Rightarrow\ z = y$ ...(1), $x - z = 1$ ...(2)

and $y - x = -1$ ...(3)

From equation (1), (2) and (3), we get

$x = 1, y = z = 0$

So $\vec{b} = x\hat{i} + y\hat{j} + z\hat{k} = \hat{i}$ $|\vec{b}| = 1$.

57. Given, $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors.

$\therefore\ |\vec{a}| = 1 = |\vec{b}| = |\vec{c}|$

Also, given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow\ |\vec{a} + \vec{b} + \vec{c}|^2 = 0 \Rightarrow\ (\vec{a} + \vec{b} + \vec{c})\cdot(\vec{a} + \vec{b} + \vec{c}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(a\cdot b+b\cdot c+c\cdot a)=0$

$\Rightarrow 1+1+1+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0 \Rightarrow \quad (\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=\frac{-3}{2}$

58. Given that $|\vec{a}+\vec{b}|=|\vec{b}|$

To prove : $(\vec{a}+2\vec{b})$ is perpendicular to $\vec{a}$.

i.e., $(\vec{a}+2\vec{b})\cdot\vec{a}=0$

Since, $|\vec{a}+\vec{b}|=|\vec{b}|$

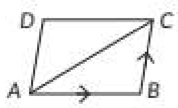
Squaring both sides, we get $|\vec{a}+\vec{b}|^2=|\vec{b}|^2$

$\Rightarrow \quad (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{b}|^2 \Rightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{b}|^2$

$\Rightarrow \quad |\vec{a}|^2+2\vec{a}\cdot\vec{b}=0 \Rightarrow \vec{a}\cdot(\vec{a}+2\vec{b})=0 \Rightarrow (\vec{a}+2\vec{b})\cdot\vec{a}=0$

$\therefore \quad \vec{a}+2\vec{b}$ is perpendicular to $\vec{a}$.

59. Given, $\overline{AB}=2\hat{i}+4\hat{j}-5\hat{k}$, $\overline{BC}=\hat{i}+2\hat{j}+3\hat{k}$



Diagonal $\overline{AC}$ of parallelogram

$ABCD=\overline{AB}+\overline{BC}$

$\overline{AC}=(2\hat{i}+4\hat{j}-5\hat{k})+(\hat{i}+2\hat{j}+3\hat{k})=3\hat{i}+6\hat{j}-2\hat{k}$

Unit vector along diagonal $\overline{AC}=\dfrac{3\hat{i}+6\hat{j}-2\hat{k}}{|3\hat{i}+6\hat{j}-2\hat{k}|}$

$=\dfrac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}}=\pm\dfrac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$

60. Here, $\vec{a}=5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b}=7\hat{i}+6\hat{j}+2\hat{k}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 6 & -2\\ 7 & 6 & 2\end{vmatrix}$$

$=\hat{i}(12+12)-\hat{j}(10+14)+\hat{k}(30-42)$

$=24\hat{i}-24\hat{j}-12\hat{k}=12(2\hat{i}-2\hat{j}-\hat{k})$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

$=\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\pm\dfrac{24\hat{i}-24\hat{j}-12\hat{k}}{\sqrt{576+576+144}}=\pm\dfrac{12(2\hat{i}-2\hat{j}-\hat{k})}{\sqrt{1296}}$

$=\pm\dfrac{12(2\hat{i}-2\hat{j}-\hat{k})}{36}=\pm\dfrac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$

61. Let $\vec{p}=|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $\vec{q}=|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$

Then we have $\vec{p}\cdot\vec{q}=\left[(|\vec{a}|\vec{b}+|\vec{b}|\vec{a})\cdot(|\vec{a}|\vec{b}-|\vec{b}|\vec{a})\right]$

$=|\vec{a}|^2(\vec{b}\cdot\vec{b})-|\vec{a}||\vec{b}|(\vec{b}\cdot\vec{a})+|\vec{b}||\vec{a}|(\vec{a}\cdot\vec{b})-|\vec{b}|^2(\vec{a}\cdot\vec{a})$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})+|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})-|\vec{b}|^2|\vec{a}|^2=0 \quad (\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a})$

$\Rightarrow \quad \vec{p}\cdot\vec{q}=0$

Hence, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$.

62. For any two non-zero vectors $\vec{a}$ and $\vec{b}$, we have

$|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$

$\Leftrightarrow \quad |\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2 \Leftrightarrow |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$

$\Leftrightarrow \quad 4\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\cdot\vec{b}=0$

So, $\vec{a}$ and $\vec{b}$ are perpendicular vectors.

63. Let $A(2\hat{i}-\hat{j}+\hat{k})$, $B(3\hat{i}+7\hat{j}+\hat{k})$ and $C(5\hat{i}+6\hat{j}+2\hat{k})$.

Then, $\overline{AB}=(3-2)\hat{i}+(7+1)\hat{j}+(1-1)\hat{k}=\hat{i}+8\hat{j}$

$\overline{AC}=(5-2)\hat{i}+(6+1)\hat{j}+(2-1)\hat{k}=3\hat{i}+7\hat{j}+\hat{k}$

$\overline{BC}=(5-3)\hat{i}+(6-7)\hat{j}+(2-1)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

Now, angle between $\overline{AC}$ and $\overline{BC}$ is given by

$\Rightarrow \quad \cos\theta=\dfrac{\overline{AC}\cdot\overline{BC}}{|\overline{AC}||\overline{BC}|}=\dfrac{6-7+1}{\sqrt{9+49+1}\sqrt{4+1+1}}$

$\Rightarrow \quad \cos\theta=0 \Rightarrow \theta=90^\circ \Rightarrow AC\perp BC$

So, *A*, *B*, *C* are the vertices of right angled triangle.

64. Given, $\hat{a}+\hat{b}=\hat{c}$

$\Rightarrow (\hat{a}+\hat{b})\cdot(\hat{a}+\hat{b})=\hat{c}\cdot\hat{c} \Rightarrow \hat{a}\cdot\hat{a}+\hat{a}\cdot\hat{b}+\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}=\hat{c}\cdot\hat{c}$

$\Rightarrow 1+\hat{a}\cdot\hat{b}+1+\hat{a}\cdot\hat{b}=1 \qquad (\because \hat{a}\cdot\hat{b}=\hat{b}\cdot\hat{a})$

$\Rightarrow 2\,\hat{a}\cdot\hat{b}=-1$ ...(i)

Now $|\hat{a}-\hat{b}|^2=(\hat{a}-\hat{b})\cdot(\hat{a}-\hat{b})$

$=\hat{a}\cdot\hat{a}-\hat{a}\cdot\hat{b}-\hat{b}\cdot\hat{a}+\hat{b}\cdot\hat{b}=1-\hat{a}\cdot\hat{b}-\hat{a}\cdot\hat{b}+1$

$=2-2\hat{a}\cdot\hat{b}=2-(-1)$ [Using (i)]

$=3$

$\therefore |\hat{a}-\hat{b}|=\sqrt{3}$

65. Given, $\vec{a}=\hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b}=3\hat{i}-\hat{j}+2\hat{k}$

Now, $\vec{a}+\vec{b}=4\hat{i}+\hat{j}-\hat{k}$

Also, $\vec{a}-\vec{b}=-2\hat{i}+3\hat{j}-5\hat{k}$

Now, $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=(4\hat{i}+\hat{j}-\hat{k})\cdot(-2\hat{i}+3\hat{j}-5\hat{k})$

$=(4)(-2)+(1)(3)+(-1)(-5)=-8+3+5=0$

Hence, $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other.

66. Here $\vec{a}=4\hat{i}-\hat{j}+8\hat{k}$ and $\vec{b}=-\hat{j}+\hat{k}$

Vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 8\\ 0 & -1 & 1\end{vmatrix}$$

$=\hat{i}(-1+8)-\hat{j}(4-0)+\hat{k}(-4-0)=7\hat{i}-4\hat{j}-4\hat{k}$

$\therefore$ Unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$=\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\pm\dfrac{7\hat{i}-4\hat{j}-4\hat{k}}{\sqrt{49+16+16}}=\pm\dfrac{1}{9}(7\hat{i}-4\hat{j}-4\hat{k})$

67. We have, $|\vec{a}|=2, |\vec{b}|=7$

$\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$

$\therefore \quad |\vec{a}\times\vec{b}|=|3\hat{i}+2\hat{j}+6\hat{k}|=\sqrt{3^2+2^2+6^2}$

$\therefore |\vec{a}\times\vec{b}|=\sqrt{49}=7$

Let 'θ' be the angle between $\vec{a}$ and $\vec{b}$, then we have

$\therefore \sin\theta=\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}=\dfrac{7}{2\times7}=\dfrac{1}{2}$

$\sin\theta=\dfrac{1}{2} \quad \therefore \theta=\dfrac{\pi}{6}$

68. Let θ be the angle between $\vec{a}$ and $\vec{b}$.

We have, $(\vec{a}\times\vec{b})^2=|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2\cdot\sin^2\theta=|\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$(\because |\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$

Hence, $(\vec{a}\times\vec{b})^2 = \vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2$

69. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

Now, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\Rightarrow (\hat{i} - 2\hat{j} + 3\hat{k})\cdot(3\hat{i} - 2\hat{j} + \hat{k}) = \sqrt{(1)^2 + (-2)^2 + (3)^2} \times \sqrt{(3)^2 + (-2)^2 + (1)^2}\cos\theta$

$\Rightarrow 3 + 4 + 3 = \sqrt{14}\times\sqrt{14}\cos\theta \Rightarrow \cos\theta = \frac{10}{14}$

$\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - \frac{100}{196}} = \sqrt{\frac{96}{196}}$

$\Rightarrow \sin\theta = \frac{4\sqrt{6}}{14} = \frac{2\sqrt{6}}{7}$

70. We have, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{a} - \vec{b} = 0\hat{i} - \hat{j} - 2\hat{k}$

A vector which is perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by

$$(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$= -2\hat{i} + 4\hat{j} - 2\hat{k} = \vec{c}$ (say)

Now, $\vec{c} = \sqrt{(-2)^2 + (4)^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$

Required unit vector, $\hat{c} = \frac{\vec{c}}{|\vec{c}|}$

$= \frac{1}{2\sqrt{6}}(-2\hat{i} + 4\hat{j} - 2\hat{k}) = -\frac{\hat{i}}{\sqrt{6}} + \frac{2\hat{j}}{\sqrt{6}} - \frac{\hat{k}}{\sqrt{6}}$

71. We have, $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0$

$\Rightarrow |\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = 0$ ... (i)

Similarly, $\vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} = 0$

$\Rightarrow \vec{a}\cdot\vec{b} + |\vec{b}|^2 + \vec{b}\cdot\vec{c} = 0$ ... (ii)

And, $\vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + \vec{c}\cdot\vec{c} = 0$

$\Rightarrow \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + |\vec{c}|^2 = 0$ ... (iii)

Adding (i), (ii) and (iii), we get

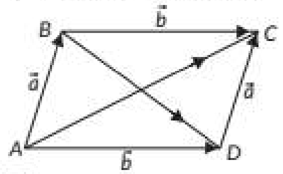
$|\vec{a}|^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{a}\cdot\vec{b} + |\vec{b}|^2 + \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + |\vec{c}|^2 = 0$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{a}\cdot\vec{c}) = 0$

$\Rightarrow (3)^2 + (4)^2 + (2)^2 + 2\mu = 0$

$\Rightarrow \mu = \frac{-(9+16+4)}{2} = \frac{-29}{2}$

72. Let $\vec{a} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$



Then diagonal $\overline{AC}$ of the parallelogram is $\vec{p} = \vec{a} + \vec{b}$

$\Rightarrow \vec{p} = 2\hat{i} - 4\hat{j} - 5\hat{k} + 2\hat{i} + 2\hat{j} + 3\hat{k} = 4\hat{i} - 2\hat{j} - 2\hat{k}$

Therefore, unit vector parallel to it is

$\frac{\vec{p}}{|\vec{p}|} = \frac{4\hat{i} - 2\hat{j} - 2\hat{k}}{\sqrt{16+4+4}} = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$

Now, diagonal $\overline{BD}$ of the parallelogram is

$\vec{p}' = \vec{b} - \vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k} - 2\hat{i} + 4\hat{j} + 5\hat{k} = 6\hat{j} + 8\hat{k}$

Therefore, unit vector parallel to it is

$\frac{\vec{p}'}{|\vec{p}'|} = \frac{6\hat{j} + 8\hat{k}}{\sqrt{36+64}} = \frac{6\hat{j} + 8\hat{k}}{10} = \frac{3\hat{j} + 4\hat{k}}{5}$

$$\text{Now, } \vec{p}\times\vec{p}' = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & 6 & 8 \end{vmatrix}$$

$= \hat{i}(-16+12) - \hat{j}(32-0) + \hat{k}(24-0)$

$= -4\hat{i} - 32\hat{j} + 24\hat{k}$

$\therefore$ Area of parallelogram $= \frac{|\vec{p}\times\vec{p}'|}{2}$

$= \frac{\sqrt{16+1024+576}}{2} = 2\sqrt{101}$ sq. units.

Concept Applied

- Area of parallelogram $= \frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$ where $\vec{d}_1$ and $\vec{d}_2$ diagonals of parallelograms.

73. Given, $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular

$\therefore \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0$ ...(i)

Also, $|\vec{a}| = |\vec{b}| = |\vec{c}|$

Let α be the angle between $(2\vec{a} + \vec{b} + 2\vec{c})$ and $\vec{a}$

$\therefore \cos\alpha = \frac{(2\vec{a} + \vec{b} + 2\vec{c})\cdot\vec{a}}{|2\vec{a} + \vec{b} + 2\vec{c}||\vec{a}|}$

$\Rightarrow \cos\alpha = \frac{2\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + 2\vec{c}\cdot\vec{a}}{|2\vec{a} + \vec{b} + 2\vec{c}||\vec{a}|}$ [From (i)]

$\cos\alpha = \frac{2|\vec{a}|^2}{|2\vec{a} + \vec{b} + 2\vec{c}||\vec{a}|} = \frac{2|\vec{a}|}{|2\vec{a} + \vec{b} + 2\vec{c}|}$ ...(ii)

Let β be the angle between $(2\vec{a} + \vec{b} + 2\vec{c})$ and $\vec{c}$

$\therefore \cos\beta = \frac{(2\vec{a} + \vec{b} + 2\vec{c})\cdot\vec{c}}{|2\vec{a} + \vec{b} + 2\vec{c}||\vec{c}|}$

$\Rightarrow \cos\beta = \frac{2\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + 2\vec{c}\cdot\vec{c}}{|2\vec{a} + \vec{b} + 2\vec{c}||\vec{c}|}$

$\Rightarrow \cos\beta = \frac{2|\vec{c}|^2}{|2\vec{a} + \vec{b} + 2\vec{c}||\vec{c}|}$ [From (i)]

$\Rightarrow \cos\beta = \frac{2|\vec{c}|}{|2\vec{a} + \vec{b} + 2\vec{c}|}$ ...(iii)

As $|\vec{a}| = |\vec{c}|$

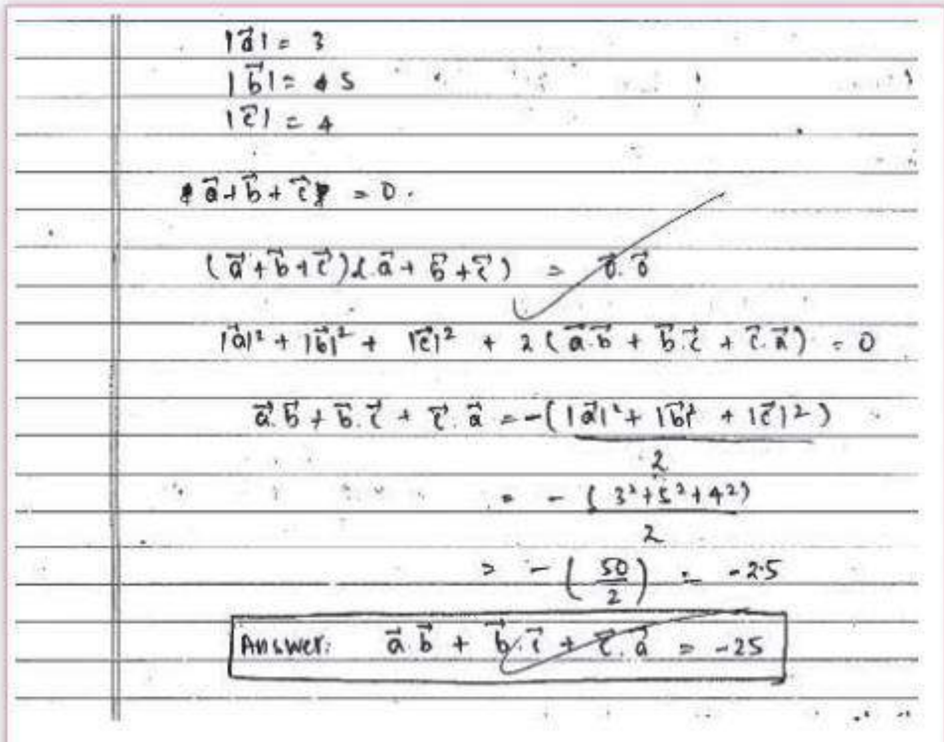
From (ii) & (iii), $\cos\alpha = \cos\beta \Rightarrow \alpha = \beta$

Hence, $(2\vec{a} + \vec{b} + 2\vec{c})$ is equally inclined to both $\vec{a}$ and $\vec{c}$.

Angle between $\vec{a}$ and $(2\vec{a} + \vec{b} + 2\vec{c})$ is

$\alpha = \cos^{-1}\left(\frac{2|\vec{a}|}{|2\vec{a} + \vec{b} + 2\vec{c}|}\right)$

74.



$|\vec{a}| = 3$

$|\vec{b}| = 5$

$|\vec{c}| = 4$

$\vec{a} + \vec{b} + \vec{c} = 0.$

$(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) = \vec{0}.\vec{0}$

$|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}) = 0$

$\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -\dfrac{(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)}{2}$

$= -\dfrac{(3^2+5^2+4^2)}{2}$

$= -\left(\dfrac{50}{2}\right) = -25$

Answer: $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -25$

[Topper's Answer, 2022]

75. We have, $|\vec{a}|=|\vec{b}|$

To prove, $\dfrac{|\vec{a}+\vec{b}|}{|\vec{a}-\vec{b}|}=\cot\dfrac{\alpha}{2}$

i.e., $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|\cot\dfrac{\alpha}{2}$

i.e., $|\vec{a}+\vec{b}|^2=|\vec{a}-\vec{b}|^2\cot^2\dfrac{\alpha}{2}$

L.H.S. $=|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}$

$=2|\vec{a}|^2+2|\vec{a}||\vec{b}|\cos\alpha \quad (\because |\vec{a}|=|\vec{b}|) = 2|\vec{a}|^2+2|\vec{a}|^2\cos\alpha$

$=2|\vec{a}|^2(1+\cos\alpha)=2|\vec{a}|^2\,2\cos^2\dfrac{\alpha}{2}$

$=4|\vec{a}|^2\cos^2\dfrac{\alpha}{2}$ ... (i)

R.H.S. $=|\vec{a}-\vec{b}|^2\cot^2\dfrac{\alpha}{2}=\left(|\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}\right)\cot^2\left(\dfrac{\alpha}{2}\right)$

$=(2|\vec{a}|^2-2|\vec{a}||\vec{b}|\cos\alpha)\cdot\cot^2\left(\dfrac{\alpha}{2}\right)$

$=2|\vec{a}|^2(1-\cos\alpha)\cot^2\dfrac{\alpha}{2}=2|\vec{a}|^2\cdot 2\sin^2\dfrac{\alpha}{2}\cdot\dfrac{\cos^2\alpha/2}{\sin^2\alpha/2}$

$=4|\vec{a}|^2\cos^2\alpha/2$ ...(ii)

$\therefore$ From (i) and (ii)
L.H.S. = R.H.S.

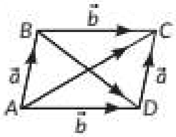
76. Let $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$

Then diagonal $\overline{AC}$ of the parallelogram is

$\vec{p}=\vec{a}+\vec{b}$

$=\hat{i}+2\hat{j}+3\hat{k}+2\hat{i}+4\hat{j}-5\hat{k}$

$=3\hat{i}+6\hat{j}-2\hat{k}$



Therefore unit vector parallel to it is

$\dfrac{\vec{p}}{|\vec{p}|}=\dfrac{3\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{9+36+4}}=\dfrac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$

Now, diagonal $\overline{BD}$ of the parallelogram is

$\vec{p}'=\vec{b}-\vec{a}=2\hat{i}+4\hat{j}-5\hat{k}-\hat{i}-2\hat{j}-3\hat{k}=\hat{i}+2\hat{j}-8\hat{k}$

Therefore unit vector parallel to it is

$\dfrac{\vec{p}'}{|\vec{p}'|}=\dfrac{\hat{i}+2\hat{j}-8\hat{k}}{\sqrt{1+4+64}}=\dfrac{1}{\sqrt{69}}(\hat{i}+2\hat{j}-8\hat{k})$

Concept Applied

- In a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$, one of the diagonal is $\vec{a}+\vec{b}$ and the other is $\vec{b}-\vec{a}$.

77. Given, ΔABC with vertices

$A(1,2,3)=\hat{i}+2\hat{j}+3\hat{k}$, $B(2,-1,4)=2\hat{i}-\hat{j}+4\hat{k}$,

$C(4,5,-1)=4\hat{i}+5\hat{j}-\hat{k}$

Now $\overline{AB}=\overline{OB}-\overline{OA}=(2\hat{i}-\hat{j}+4\hat{k})-(\hat{i}+2\hat{j}+3\hat{k}) = \hat{i}-3\hat{j}+\hat{k}$.

$\overline{AC}=\overline{OC}-\overline{OA}=(4\hat{i}+5\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+3\hat{k}) = 3\hat{i}+3\hat{j}-4\hat{k}$.

$\therefore \quad (\overline{AB}\times\overline{AC})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}=9\hat{i}+7\hat{j}+12\hat{k}$

Hence, area of $\Delta ABC=\dfrac{1}{2}|\overline{AB}\times\overline{AC}| = \dfrac{1}{2}|9\hat{i}+7\hat{j}+12\hat{k}|$

$=\dfrac{1}{2}\sqrt{9^2+7^2+12^2}=\dfrac{1}{2}\sqrt{81+49+144} = \dfrac{1}{2}\sqrt{274}$ sq. units

78. Since $\vec{a},\vec{b}$ and $\vec{c}$ are the position vector of A, B and C respectively.

Then $\overline{BC}$ = position vector of C – position vector of B

$= \vec{c} - \vec{b}$ ...(i)

and $\overline{CA}$ = position vector of A – position vector of C

$= \vec{a} - \vec{c}$ ...(ii)

A, B and C are collinear if and only if $\overline{BC} \times \overline{CA} = \vec{0}$

if and only if $(\vec{c} - \vec{b}) \times (\vec{a} - \vec{c}) = \vec{0}$ {From (i) and (ii)}

if and only if $(\vec{c} \times \vec{a}) - (\vec{c} \times \vec{c}) - (\vec{b} \times \vec{a}) + (\vec{b} \times \vec{c}) = \vec{0}$

if and only if $(\vec{c} \times \vec{a}) + (\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) = \vec{0}$

$\{\because \vec{c} \times \vec{c} = \vec{0} \text{ and } \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}\}$

iff $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = \vec{0}$

$\therefore$ A, B and C are collinear iff $(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a}) = \vec{0}$

79. Here, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$\Rightarrow \vec{b} + \vec{c} = (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

The unit vector along $\vec{b} + \vec{c}$ is $\vec{p} = \dfrac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}$

$= \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}} = \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}}$

Also, $\vec{a}.\vec{p} = 1$ (Given)

$\Rightarrow \dfrac{(2+\lambda) + 6 - 2}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1 \Rightarrow \sqrt{\lambda^2 + 4\lambda + 44} = \lambda + 6$

$\Rightarrow \lambda^2 + 4\lambda + 44 = \lambda^2 + 12\lambda + 36 \Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1$

$\therefore$ The required unit vector

$\vec{p} = \dfrac{(2+1)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{1 + 4 + 44}} = \dfrac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$.

80. Given, position vector of $A = \hat{i} + \hat{j} + \hat{k}$

Position vector of $B = 2\hat{i} + 5\hat{j}$

Position vector of $C = 3\hat{i} + 2\hat{j} - 3\hat{k}$

Position vector of $D = \hat{i} - 6\hat{j} - \hat{k}$

$\therefore \overline{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$

and $\overline{CD} = (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{i} - 8\hat{j} + 2\hat{k}$

Now, $|\overline{AB}| = \sqrt{(1)^2 + (4)^2 + (-1)^2} = \sqrt{18}$

$|\overline{CD}| = \sqrt{(-2)^2 + (-8)^2 + (2)^2} = \sqrt{4 + 64 + 4} = \sqrt{72} = 2\sqrt{18}$

$= \sqrt{72} = 2\sqrt{18}$

Let θ be the angle between $\overline{AB}$ and $\overline{CD}$.

$\therefore \cos\theta = \dfrac{\overline{AB} \cdot \overline{CD}}{|\overline{AB}||\overline{CD}|} = \dfrac{(\hat{i} + 4\hat{j} - \hat{k}) \cdot (-2\hat{i} - 8\hat{j} + 2\hat{k})}{(\sqrt{18})(2\sqrt{18})}$

$= \dfrac{-2 - 32 - 2}{36} = \dfrac{-36}{36} = -1 \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

Since, angle between $\overline{AB}$ and $\overline{CD}$ is 180°.

$\therefore$ $\overline{AB}$ and $\overline{CD}$ are collinear.

81. We have, $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $|\vec{c}| = 3$ ...(i)

Given, Projection of $\vec{b}$ along $\vec{a}$ = Projection of $\vec{c}$ along $\vec{a}$

$\Rightarrow \dfrac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \dfrac{\vec{c} \cdot \vec{a}}{|\vec{a}|} \Rightarrow \vec{b} \cdot \vec{a} = \vec{c} \cdot \vec{a}$

$\Rightarrow \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ ...(ii)

Given, $\vec{b}$ and $\vec{c}$ are perpendicular to each other

$\therefore \vec{b} \cdot \vec{c} = 0$ ...(iii)

Now, $|3\vec{a} - 2\vec{b} + 2\vec{c}|^2 = (3\vec{a} - 2\vec{b} + 2\vec{c}) \cdot (3\vec{a} - 2\vec{b} + 2\vec{c})$

$= 9(\vec{a} \cdot \vec{a}) - 6(\vec{a} \cdot \vec{b}) + 6(\vec{a} \cdot \vec{c}) - 6(\vec{b} \cdot \vec{a}) + 4(\vec{b} \cdot \vec{b}) - 4(\vec{b} \cdot \vec{c})$

$+ 6(\vec{c} \cdot \vec{a}) - 4(\vec{c} \cdot \vec{b}) + 4(\vec{c} \cdot \vec{c})$

$= 9|\vec{a}|^2 + 4|\vec{b}|^2 + 4|\vec{c}|^2 + 2\{-6(\vec{a} \cdot \vec{b}) - 4(\vec{b} \cdot \vec{c}) + 6(\vec{a} \cdot \vec{c})\}$

$= 9 \times 1^2 + 4 \times 2^2 + 4 \times 3^2 + 2\{-6(\vec{a} \cdot \vec{b}) - 4 \times 0 + 6(\vec{a} \cdot \vec{b})\}$

[From eqn (i) and (iii)]

$= 9 + 16 + 36 + 2 \times 0 = 61$ [From eqn. (ii)]

$\therefore |3\vec{a} - 2\vec{b} + 2\vec{c}| = \sqrt{61}$

82. Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$

Now, it is given that, $\vec{d}$ is perpendicular to

$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$ $\therefore \vec{d} \cdot \vec{b} = 0$ and $\vec{d} \cdot \vec{c} = 0$

$\Rightarrow x - 4y + 5z = 0$ ...(i)

and $3x + y - z = 0$ ...(ii)

Also, $\vec{d} \cdot \vec{a} = 21$, where $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$

$\Rightarrow 4x + 5y - z = 21$ ...(iii)

Eliminating z from (i) and (ii), we get $16x + y = 0$ ...(iv)

Eliminating z from (ii) and (iii), we get $x + 4y = 21$ ...(v)

Solving (iv) and (v), we get $x = \dfrac{-1}{3}, y = \dfrac{16}{3}$

Putting the values of x and y in (i), we get $z = \dfrac{13}{3}$

$\therefore \vec{d} = \dfrac{-1}{3}\hat{i} + \dfrac{16}{3}\hat{j} + \dfrac{13}{3}\hat{k}$ is the required vector.

Concept Applied

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$.

83. Given, $|\vec{a}| = |\vec{b}| = |\vec{c}|$ ...(i)

and $\vec{a} \cdot \vec{b} = 0, \vec{b} \cdot \vec{c} = 0, \vec{c} \cdot \vec{a} = 0$ ...(ii)

Let $(\vec{a} + \vec{b} + \vec{c})$ be inclined to vectors $\vec{a}, \vec{b}$ and $\vec{c}$ by angles α, β and γ respectively. Then

$\cos\alpha = \dfrac{(\vec{a} + \vec{b} + \vec{c}) \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|} = \dfrac{\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|}$

$= \dfrac{|\vec{a}|^2 + 0 + 0}{|\vec{a} + \vec{b} + \vec{c}||\vec{a}|}$ [Using (ii)]

$= \dfrac{|\vec{a}|}{|\vec{a} + \vec{b} + \vec{c}|}$ ...(iii)

Similarly, $\cos\beta = \dfrac{|\vec{b}|}{|\vec{a} + \vec{b} + \vec{c}|}$ ...(iv)

and $\cos\gamma = \dfrac{|\vec{c}|}{|\vec{a} + \vec{b} + \vec{c}|}$ ...(v)

From (i), (iii), (iv) and (v), we get

$\cos\alpha = \cos\beta = \cos\gamma \Rightarrow \alpha = \beta = \gamma$

Hence, the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to the vector $\vec{a}, \vec{b}$ and $\vec{c}$.

Also, the angle between them is given as

$\alpha=\cos^{-1}\left(\frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}\right), \beta=\cos^{-1}\left(\frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$

$\gamma=\cos^{-1}\left(\frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}\right)$

84. We have, $A(2\hat{i}-\hat{j}+\hat{k}), B(\hat{i}-3\hat{j}-5\hat{k})$,

and $C(3\hat{i}-4\hat{j}-4\hat{k})$

Then, $\overline{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$,

$\overline{AC}=(3-2)\hat{i}+(-4+1)\hat{j}+(-4-1)\hat{k}=\hat{i}-3\hat{j}-5\hat{k}$

and $\overline{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$

Now angle between $\overline{AC}$ and $\overline{BC}$ is given by

$\cos\theta=\frac{(\overline{AC})\cdot(\overline{BC})}{|\overline{AC}||\overline{BC}|}=\frac{2+3-5}{\sqrt{1+9+25}.\sqrt{4+1+1}}$

$\Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2} \Rightarrow BC\perp AC$

So, A, B, C are vertices of right angled triangle.

Now area of $\Delta ABC=\frac{1}{2}|\overline{AC}\times\overline{BC}|$

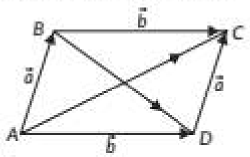
$=\frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & -5\\ 2 & -1 & 1\end{matrix}\right\|=\frac{1}{2}|(-3-5)\hat{i}-(1+10)\hat{j}+(-1+6)\hat{k}|$

$=\frac{1}{2}|-8\hat{i}-11\hat{j}+5\hat{k}|=\frac{1}{2}\sqrt{64+121+25}=\frac{\sqrt{210}}{2}$ sq. units.

Concept Applied

- If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$, then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

85. Let $\vec{a}=2\hat{i}-4\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+2\hat{j}+3\hat{k}$



Then diagonal $\overline{AC}$ of the parallelogram is $\vec{p}=\vec{a}+\vec{b}$

$\Rightarrow \vec{p}=2\hat{i}-4\hat{j}-5\hat{k}+2\hat{i}+2\hat{j}+3\hat{k}=4\hat{i}-2$

Therefore, unit vector parallel to it is

$\frac{\vec{p}}{|\vec{p}|}=\frac{4\hat{i}-2\hat{j}-2\hat{k}}{\sqrt{16+4+4}}=\frac{2\hat{i}-\hat{j}-\hat{k}}{\sqrt{6}}$

Now, diagonal $\overline{BD}$ of the parallelogram is

$\vec{p}'=\vec{b}-\vec{a}=2\hat{i}+2\hat{j}+3\hat{k}-2\hat{i}+4\hat{j}+5\hat{k}=6\hat{j}+8\hat{k}$

Therefore, unit vector parallel to it is

$\frac{\vec{p}'}{|\vec{p}'|}=\frac{6\hat{j}+8\hat{k}}{\sqrt{36+64}}=\frac{6\hat{j}+8\hat{k}}{10}=\frac{3\hat{j}+4\hat{k}}{5}$

Now, $\vec{p}\times\vec{p}'=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -2 & -2\\ 0 & 6 & 8\end{vmatrix}$

$=\hat{i}(-16+12)-\hat{j}(32-0)+\hat{k}(24-0)=-4\hat{i}-32\hat{j}+24\hat{k}$

$\therefore$ Area of parallelogram $=\frac{|\vec{p}\times\vec{p}'|}{2}$

$=\frac{\sqrt{16+1024+576}}{2}=2\sqrt{101}$ sq. units.

Concept Applied

- Area of parallelogram $=\frac{1}{2}|\vec{d}_1\times\vec{d}_2|$ where $\vec{d}_1$ and $\vec{d}_2$ diagonals of parallelograms.

86. Two non zero vectors are parallel if and only if their cross product is zero vector.

So, we have to prove that cross product of $\vec{a}-\vec{d}$ and $\vec{b}-\vec{c}$ is zero vector.

$(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})-(\vec{d}\times\vec{b})+(\vec{d}\times\vec{c})$

Since, it is given that $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$.

And, $\vec{d}\times\vec{b}=-\vec{b}\times\vec{d}$, $\vec{d}\times\vec{c}=-\vec{c}\times\vec{d}$

Therefore, $(\vec{a}-\vec{d})\times(\vec{b}-\vec{c})=(\vec{c}\times\vec{d})-(\vec{b}\times\vec{d})+(\vec{b}\times\vec{d})-(\vec{c}\times\vec{d})=\vec{0}$

Hence, $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.

87. $(\vec{r}\times\hat{i})\cdot(\vec{r}\times\hat{j})+xy=[(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{i}]\cdot[(x\hat{i}+y\hat{j}+z\hat{k})\times\hat{j})]+xy$

$=(-y\hat{k}+z\hat{j})\cdot(x\hat{k}-z\hat{i})+xy=-xy+xy=0$

88. Here, $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}$ and $\vec{c}=3\hat{i}-4\hat{j}-5\hat{k}$

$\therefore \vec{a}-\vec{b}=(\hat{i}+2\hat{j}+\hat{k})-(2\hat{i}+\hat{j})=-\hat{i}+\hat{j}+\hat{k}$

and $\vec{c}-\vec{b}=(3\hat{i}-4\hat{j}-5\hat{k})-(2\hat{i}+\hat{j})=\hat{i}-5\hat{j}-5\hat{k}$

Vector perpendicular to both $\vec{a}-\vec{b}$ and $\vec{c}-\vec{b}$ is

$(\vec{a}-\vec{b})\times(\vec{c}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 1 & 1\\ 1 & -5 & -5\end{vmatrix}$

$=(-5+5)\hat{i}-(5-1)\hat{j}+(5-1)\hat{k}=-4\hat{j}+4\hat{k}$

$\therefore$ Unit vector perpendicular to both $\vec{a}-\vec{b}$ and $\vec{c}-\vec{b}$

$=\pm\frac{-4\hat{j}+4\hat{k}}{|-4\hat{j}+4\hat{k}|}=\pm\frac{-4\hat{j}+4\hat{k}}{\sqrt{(-4)^2+4^2}}=\pm\frac{-4\hat{j}+4\hat{k}}{4\sqrt{2}}=\pm\frac{1}{\sqrt{2}}(-\hat{j}+\hat{k})$.

89. Given $\vec{a}+\vec{b}+\vec{c}=0$ and $|\vec{a}|=3, |\vec{b}|=5, |\vec{c}|=7$

We have $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c} \Rightarrow |\vec{a}+\vec{b}|^2=|-\vec{c}|^2 \Rightarrow |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=|\vec{c}|^2$

$\Rightarrow 9+25+2|\vec{a}||\vec{b}|\cos\theta=49 \Rightarrow 2\times3\times5\times\cos\theta=49-34=15$

$\Rightarrow \cos\theta=\frac{15}{30}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}=60^\circ$

90. We have, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Let $\vec{r}=\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{p}=\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A unit vector perpendicular to both $\vec{r}$ and $\vec{p}$ is given as $\pm\frac{\vec{r}\times\vec{p}}{|\vec{r}\times\vec{p}|}$.

Now, $\vec{r}\times\vec{p}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}=-2\hat{i}+4\hat{j}-2\hat{k}$

So, the required unit vector is

$=\pm\frac{(-2\hat{i}+4\hat{j}-2\hat{k})}{\sqrt{(-2)^2+4^2+(-2)^2}}=\mp\frac{(\hat{i}-2\hat{j}+\hat{k})}{\sqrt{6}}$.

91. Here, $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+\hat{k}$ and $\vec{c}=2\hat{j}-\hat{k}$

$\therefore\ \vec{a}+\vec{b}=(2\hat{i}-3\hat{j}+\hat{k})+(-\hat{i}+\hat{k})=\hat{i}-3\hat{j}+2\hat{k}$

and $\vec{b}+\vec{c}=(-\hat{i}+\hat{k})+(2\hat{j}-\hat{k})=-\hat{i}+2\hat{j}$

$$\therefore (\vec{a}+\vec{b})\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 2\\ -1 & 2 & 0\end{vmatrix}=-4\hat{i}-2\hat{j}-\hat{k}$$

$\therefore$ Area of a parallelogram whose diagonals are $\vec{a}+\vec{b}$ and $\vec{b}+\vec{c}$

$=\frac{1}{2}|(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})|=\frac{1}{2}|-4\hat{i}-2\hat{j}-\hat{k}|$

$=\frac{1}{2}\sqrt{(-4)^2+(-2)^2+(-1)^2}=\frac{\sqrt{21}}{2}$ sq.units.

92. Let $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$

Now, it is given that $\vec{p}$ is perpendicular to both $\vec{\alpha}$ and $\vec{\beta}$

$\therefore\quad \vec{p}\cdot\vec{\alpha}=0$ and $\vec{p}\cdot\vec{\beta}=0$

$\Rightarrow\quad (x\hat{i}+y\hat{j}+z\hat{k})\cdot(4\hat{i}+5\hat{j}-\hat{k})=0$

and $(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-4\hat{j}+5\hat{k})=0$

$\Rightarrow\quad 4x+5y-z=0$ ...(i)

and $x-4y+5z=0$ ...(ii)

Also, we have, $\vec{p}\cdot\vec{q}=21\Rightarrow(x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}+\hat{j}-\hat{k})=21$

$\Rightarrow\quad 3x+y-z=21$ ...(iii)

Eliminating z from (i) and (ii), we get

$21x+21y=0\Rightarrow x+y=0$ ...(iv)

Eliminating z from (i) and (iii), we get $x+4y=-21$ ...(v)

Solving (iv) and (v), we get $x=7, y=-7$

Now, from (i), we get $z=-7$

So, $\vec{p}=7\hat{i}-7\hat{j}-7\hat{k}$.

CBSE Sample Questions

1. Let $\vec{a}$ be the unit vector in the direction opposite to the given vector $\left(-\frac{3}{4}\hat{j}\right)$.

Then, $\vec{a}=\frac{-1}{\sqrt{\left(\frac{3}{4}\right)^2}}\left(-\frac{3}{4}\hat{j}\right)=\hat{j}$ (1)

2. A vector in the direction opposite to $2\hat{i}+3\hat{j}-6\hat{k}$ is $-2\hat{i}-3\hat{j}+6\hat{k}$.

Its magnitude is $|\sqrt{4+9+36}|=|\sqrt{49}|=7$

So, a vector in the direction opposite to $2\hat{i}+3\hat{j}-6\hat{k}$ of magnitude 5 units is $\frac{5}{7}(-2\hat{i}-3\hat{j}+6\hat{k})$ (1)

3. (a): Scalar projection of $\vec{a}$ on $\vec{b}=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Scalar projection of $3\hat{i}-\hat{j}-2\hat{k}$ on vector $\hat{i}+2\hat{j}-3\hat{k}$

$=\frac{(3\hat{i}-\hat{j}-2\hat{k})\cdot(\hat{i}+2\hat{j}-3\hat{k})}{|\hat{i}+2\hat{j}-3\hat{k}|}=\frac{7}{\sqrt{14}}$ (1)

4. (b): $|\vec{a}-2\vec{b}|^2=(\vec{a}-2\vec{b})\cdot(\vec{a}-2\vec{b})$

$=\vec{a}\cdot\vec{a}-4\vec{a}\cdot\vec{b}+4\vec{b}\cdot\vec{b}=|\vec{a}|^2-4\vec{a}\cdot\vec{b}+4|\vec{b}|^2=4-16+36=24$

$\therefore\ |\vec{a}-2\vec{b}|=2\sqrt{6}$ (1)

5. Area of the triangle

$=\frac{1}{2}|2\hat{i}\times(-3)\hat{j}|=\frac{1}{2}|-6\hat{k}|=3$ sq. units (1)

6. We have, $|\hat{a}+\hat{b}|^2=1\Rightarrow\hat{a}^2+\hat{b}^2+2\hat{a}\cdot\hat{b}=1$

$\Rightarrow\quad 2\hat{a}\cdot\hat{b}=1-1-1$ $(\because|\hat{a}|=|\hat{b}|=1)$

$\Rightarrow\ \hat{a}\cdot\hat{b}=\frac{-1}{2}\Rightarrow|\hat{a}||\hat{b}|\cos\theta=\frac{-1}{2}\Rightarrow\theta=\cos^{-1}\left(-\frac{1}{2}\right)$

$\Rightarrow\ \theta=\pi-\frac{\pi}{3}\Rightarrow\theta=\frac{2\pi}{3}$ (1)

7. Since $\vec{a}$ is a unit vector, $\therefore\ |\vec{a}|=1$

Now, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$

$\Rightarrow\ \vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=12$ (1/2)

$\Rightarrow|\vec{x}|^2-|\vec{a}|^2=12$ $(\because\vec{a}\cdot\vec{x}=\vec{x}\cdot\vec{a})$ (1/2)

$\Rightarrow\ |\vec{x}|^2-1=12\Rightarrow|\vec{x}|^2=13\Rightarrow|\vec{x}|=\sqrt{13}$ (1)

8. Since, $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})$ (1)

$\therefore\ |\vec{a}+\vec{b}|^2=1+1+2\cos\theta$ [As $|\vec{a}|=|\vec{b}|=1$]

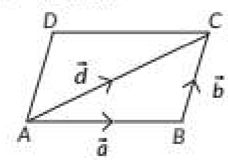
$=2(1+\cos\theta)=4\cos^2\frac{\theta}{2}$ $\left[\because 1+\cos\theta=2\cos^2\frac{\theta}{2}\right]$ (1/2)

$\therefore\ |\vec{a}+\vec{b}|=2\cos\frac{\theta}{2}$ (1/2)

9. Let $ABCD$ is a parallelogram such that

$\vec{a}=\overline{AB}=\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=\overline{BC}$, and $\vec{d}=\overline{AC}=4\hat{i}+5\hat{k}$.

Now, $\vec{a}+\vec{b}=\vec{d}$ (By triangle law)



$\Rightarrow\ \vec{b}=\vec{d}-\vec{a}$

$\Rightarrow\ \vec{b}=(4\hat{i}+5\hat{k})-(\hat{i}-\hat{j}+\hat{k})$

$=3\hat{i}+\hat{j}+4\hat{k}$ (1/2)

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 1\\ 3 & 1 & 4\end{vmatrix}=-5\hat{i}-\hat{j}+4\hat{k}$ (1)

$\therefore$ Area of parallelogram $=|\vec{a}\times\vec{b}|$

$=\sqrt{25+1+16}=\sqrt{42}$ sq.units (1/2)

10. We have, $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}\Rightarrow\vec{a}\cdot(\vec{b}-\vec{c})=0$

$\Rightarrow\quad(\vec{b}-\vec{c})=\vec{0}$ or $\vec{a}\perp(\vec{b}-\vec{c})$

$\Rightarrow\quad\vec{b}=\vec{c}$ or $\vec{a}\perp(\vec{b}-\vec{c})$ (1)

Also, $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}\Rightarrow\vec{a}\times(\vec{b}-\vec{c})=\vec{0}$

$\Rightarrow\quad\vec{b}-\vec{c}=\vec{0}$ or $\vec{a}\,||\,(\vec{b}-\vec{c})$

$\Rightarrow\quad\vec{b}=\vec{c}$ or $\vec{a}\,||\,(\vec{b}-\vec{c})$ (1)

Since, $\vec{a}$ can not be both perpendicular to $(\vec{b}-\vec{c})$ and parallel to $(\vec{b}-\vec{c})$.

Hence, $\vec{b}=\vec{c}$. (1)