

Inverse Trigonometric Functions

1 Mark Questions

1. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x . Delhi 2014

$$\text{Given, } \sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}(1)$$

$$[\because \sin \theta = x \Rightarrow \theta = \sin^{-1}x]$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}\left(\sin\frac{\pi}{2}\right) \left[\because \sin\frac{\pi}{2} = 1\right]$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2} \quad (1/2)$$

$$\text{But we know that, } \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2},$$

$$x \in [-1, 1]$$

$$\therefore \sin^{-1}\frac{1}{5} = \sin^{-1}x \Rightarrow x = \frac{1}{5} \quad (1/2)$$

2. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$; $xy < 1$, then write the value of $x + y + xy$. All India 2014

Given, $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$, $xy < 1$

We know that,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left[\frac{x+y}{1-xy} \right], xy < 1$$

$$\therefore \tan^{-1} \left[\frac{x+y}{1-xy} \right] = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy} = \tan \frac{\pi}{4} \quad (1/2)$$

$$\Rightarrow x+y = 1-xy \quad \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\Rightarrow x+y+xy = 1 \quad (1/2)$$

3. Write the value of $\cos^{-1} \left(-\frac{1}{2} \right) + 2\sin^{-1} \left(\frac{1}{2} \right)$.


Foreign 2014

$$\begin{aligned}
 &\text{We have, } \cos^{-1}\left(\frac{-1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right) \\
 &= \left[\pi - \cos^{-1}\left(\frac{1}{2}\right) \right] + 2 \sin^{-1}\left(\frac{1}{2}\right) \\
 &\qquad\qquad\qquad [\because \cos^{-1}(-x) = \pi - \cos^{-1} x] \\
 &= \left[\pi - \cos^{-1}\left(\cos \frac{\pi}{3}\right) \right] + 2 \sin^{-1}\left(\sin \frac{\pi}{6}\right) \quad (1/2)
 \end{aligned}$$

$[\because \text{principal value branch of } \cos^{-1} x \text{ is } [0, \pi]$
and that of $\sin^{-1} x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]]$

$$\begin{aligned}
 &= \left[\pi - \frac{\pi}{3} \right] + 2 \times \frac{\pi}{6} \\
 &= \frac{2\pi}{3} + \frac{2\pi}{6} = \frac{4\pi + 2\pi}{6} = \pi \qquad\qquad\qquad (1/2)
 \end{aligned}$$

4. Write the principal value of $\cos^{-1}[\cos(680)^\circ]$.
Delhi 2014C

 Firstly, write 680° in the form of $(2\pi - \theta)$ and then use $\cos(2\pi - \theta) = \cos\theta$ and $\cos^{-1}(\cos\theta) = \theta$.

$$\begin{aligned}
 &\text{We have, } \cos^{-1}[\cos(680)^\circ] \\
 &= \cos^{-1}[\cos(2 \times 360 - 40)^\circ] \\
 &= \cos^{-1}[\cos 40^\circ] \qquad\qquad [\because \cos(2\pi - \theta) = \cos\theta] \\
 &= 40^\circ \qquad\qquad\qquad [\because \cos^{-1}(\cos\theta) = \theta] \quad (1)
 \end{aligned}$$

5. Write the principal value of $\tan^{-1}\left[\sin\left(\frac{-\pi}{2}\right)\right]$.
All India 2014C


$$\begin{aligned} \text{We have, } \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] \\ = \tan^{-1}[-1] \quad \left[\because \sin\left(-\frac{\pi}{2}\right) = -1\right] \end{aligned}$$

\therefore Principal value branch of \tan^{-1} is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\begin{aligned} \therefore \tan^{-1}(-1) &= \tan^{-1}\left(-\tan\frac{\pi}{4}\right) \quad \left[\because \tan\frac{\pi}{4} = 1\right] \\ &= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] = -\frac{\pi}{4} \quad \text{(1) } [\tan^{-1}(\tan\theta) = \theta] \end{aligned}$$

6. Find the value of the following:

$$\cot\left(\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right) \quad \text{All India 2014 C}$$

 Firstly, use the property $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$, then put $\cot^{-1}\sqrt{3} = \frac{\pi}{6}$ and simplify.

$$\begin{aligned} \text{We have, } \cot\left[\frac{\pi}{2} - 2\cot^{-1}\sqrt{3}\right] \\ = \tan(2\cot^{-1}\sqrt{3}) \quad \left[\because \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta\right] \quad \text{(1/2)} \\ = \tan\left(2 \times \frac{\pi}{6}\right) \quad \left[\because \cot^{-1}\sqrt{3} = \cot^{-1}\left(\cot\frac{\pi}{6}\right) = \frac{\pi}{6}\right] \\ = \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \quad \text{(1/2)} \end{aligned}$$

7. Write the principal value of

$$\left[\cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\left(-\frac{1}{2}\right)\right] \quad \text{Delhi 2013C}$$

We have, $\cos^{-1} \frac{\sqrt{3}}{2} + \cos^{-1} \left(-\frac{1}{2} \right)$

$$= \cos^{-1} \frac{\sqrt{3}}{2} + \left[\pi - \cos^{-1} \left(\frac{1}{2} \right) \right]$$

$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$

$$= \frac{\pi}{6} + \pi - \frac{\pi}{3} = \frac{\pi + 6\pi - 2\pi}{6} = \frac{5\pi}{6} \quad (1)$$

8. Write the value of $\tan^{-1} \left(\frac{a}{b} \right) - \tan^{-1} \left(\frac{a-b}{a+b} \right)$.

Delhi 2013C

$$\tan^{-1} \left(\frac{a}{b} \right) - \tan^{-1} \left(\frac{a-b}{a+b} \right) = \tan^{-1} \frac{\left[\frac{a}{b} - \frac{(a-b)}{a+b} \right]}{1 + \frac{a}{b} \frac{(a-b)}{a+b}}$$

$$\left[\because \tan^{-1} A - \tan^{-1} B = \tan^{-1} \left(\frac{A-B}{1+AB} \right) \right]$$


$$= \tan^{-1} \left(\frac{a^2 + ab - ab + b^2}{ab + b^2 + a^2 - ab} \right) \quad (1/2)$$

$$= \tan^{-1} \left(\frac{a^2 + b^2}{a^2 + b^2} \right) = \tan^{-1} 1$$

$$= \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4} \quad [\because \tan^{-1}(\tan \theta) = \theta] \quad (1/2)$$

9. Write the principal value of

$$\tan^{-1}(1) + \cos^{-1} \left(-\frac{1}{2} \right) \quad \text{HOTS; Delhi 2013}$$

 Firstly, we check the given angle in principal value branch. If it is not so, then convert it. After that, use the identity $\tan^{-1}(\tan\theta) = \theta$, $\cos^{-1}(\cos\theta) = \theta$.

$$\begin{aligned}
 & \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) \\
 &= \tan^{-1}\left(\tan\frac{\pi}{4}\right) + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) \quad (1/2) \\
 &= \frac{\pi}{4} + \cos^{-1}\left[\cos\left(\pi - \frac{\pi}{3}\right)\right]
 \end{aligned}$$

[since, principal value branch of $\cos^{-1}x$ is $[0, \pi]$, so we convert $-\cos\theta = \cos(\pi - \theta)$]

$$\begin{aligned}
 &= \frac{\pi}{4} + \cos^{-1}\left[\cos\frac{2\pi}{3}\right] \\
 &= \frac{\pi}{4} + \frac{2\pi}{3} = \frac{3\pi + 8\pi}{12} \\
 &= \frac{11\pi}{12} \quad (1/2)
 \end{aligned}$$

10. Write the value of $\tan\left(2 \tan^{-1} \frac{1}{5}\right)$. Delhi 2013

$$\tan\left(2 \tan^{-1} \frac{1}{5}\right) = \tan\left[\tan^{-1}\left(\frac{2 \times \frac{1}{5}}{1 - \left(\frac{1}{5}\right)^2}\right)\right] \quad (1/2)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{2 \times 5}{24}\right)\right] = \tan\left[\tan^{-1}\left(\frac{5}{12}\right)\right] = \frac{5}{12}$$

$$\left[\because \tan(\tan^{-1} \theta) = \theta\right] \quad (1/2)$$

11. Write the value of $\tan^{-1} \left[2 \sin \left(2 \cos^{-1} \frac{\sqrt{3}}{2} \right) \right]$.

All India 2013

$$\begin{aligned} & \tan^{-1} \left[2 \sin \left(2 \cos^{-1} \frac{\sqrt{3}}{2} \right) \right] \\ &= \tan^{-1} \left[2 \sin \left\{ \cos^{-1} \left(2 \cdot \frac{3}{4} - 1 \right) \right\} \right] \\ & \quad [\because 2 \cos^{-1} x = \cos^{-1} (2x^2 - 1)] \\ &= \tan^{-1} \left[2 \sin \left\{ \cos^{-1} \left(\frac{3}{2} - 1 \right) \right\} \right] \\ &= \tan^{-1} \left[2 \sin \left\{ \cos^{-1} \left(\frac{1}{2} \right) \right\} \right] \quad (1/2) \\ &= \tan^{-1} \left[2 \sin \left\{ \cos^{-1} \left(\cos \frac{\pi}{3} \right) \right\} \right] \\ &= \tan^{-1} \left[2 \sin \frac{\pi}{3} \right] [\because \cos^{-1}(\cos \theta) = \theta] \\ &= \tan^{-1} \left(2 \cdot \frac{\sqrt{3}}{2} \right) \\ &= \tan^{-1} (\sqrt{3}) = \tan^{-1} \left(\tan \frac{\pi}{3} \right) = \frac{\pi}{3} \\ & \quad [\because \tan^{-1}(\tan \theta) = \theta] \quad (1/2) \end{aligned}$$

12. Write the principal value of

$$\tan^{-1} (\sqrt{3}) - \cot^{-1} (-\sqrt{3}). \quad \text{All India 2013}$$

$$\begin{aligned}
& \tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}) \\
&= \tan^{-1}(\sqrt{3}) - \{\pi - \cot^{-1}(\sqrt{3})\} \\
& \left[\begin{array}{l} \because \text{principal value branch of } \cot^{-1}x \text{ is } (0, \pi) \\ \because \cot^{-1}(-x) = \pi - \cot^{-1}x \end{array} \right] \\
&= \tan^{-1}\sqrt{3} - \pi + \cot^{-1}\sqrt{3} \\
&= (\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}) - \pi \\
&= \frac{\pi}{2} - \pi = -\frac{\pi}{2} \left[\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2} \right] \quad (1)
\end{aligned}$$

13. Write the value of $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$.

Delhi 2012

? Firstly, we check the given angle in principal value branch, then use the identity $\sin^{-1}(\sin \theta) = \theta$ and $\cos(\cos^{-1} \theta) = \theta$.

$$\begin{aligned}
& \cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right) \\
&= \cos^{-1}\left(\cos\frac{\pi}{3}\right) - 2\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right] \\
& \left[\begin{array}{l} \because \text{principal value branch of } \cos^{-1}x \\ \text{is } [0, \pi] \text{ and that of } \sin^{-1} \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{array} \right] \\
&= \frac{\pi}{3} - 2\left(-\frac{\pi}{6}\right) \\
& \left[\because \cos^{-1}(\cos \theta) = \theta \text{ and } \sin^{-1}(\sin \theta) = \theta \right] \\
&= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3} \quad (1)
\end{aligned}$$

14. Find the principal value of

$$\tan^{-1} \sqrt{3} - \sec^{-1} (-2).$$

All India 2012



Given expression is not standard identity, so we separately find the value of $\tan^{-1}(\sqrt{3})$ and $\sec^{-1}(-2)$, then simplify it.

We know that, the principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and that of \sec^{-1} is

$$[0, \pi] - \left\{\frac{\pi}{2}\right\}.$$

$$\therefore \tan^{-1} \sqrt{3} - \sec^{-1} (-2)$$

$$= \tan^{-1} \left(\tan \frac{\pi}{3} \right) - \sec^{-1} \left(\sec \frac{2\pi}{3} \right)$$

$$\left[\because \tan \frac{\pi}{3} = \sqrt{3} \text{ and } \sec \frac{2\pi}{3} = -2 \right]$$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = \frac{-\pi}{3}$$

$$[\because \tan^{-1} (\tan \theta) = \theta \text{ and } \sec^{-1} (\sec \theta) = \theta] \quad (1)$$

15. Using the principal values, write the value of

$$\cos^{-1} \left(\frac{1}{2} \right) + 2 \sin^{-1} \left(\frac{1}{2} \right).$$


All India 2012C

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$ and of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned} \therefore \cos^{-1}\left(\frac{1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right) &= \cos^{-1}\left(\cos \frac{\pi}{3}\right) + 2 \sin^{-1}\left(\sin \frac{\pi}{6}\right) \\ &\left[\because \cos \frac{\pi}{3} = \frac{1}{2} \text{ and } \sin \frac{\pi}{6} = \frac{1}{2} \right] \\ &= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3} \\ &[\because \cos^{-1}(\cos \theta) = \theta \text{ and } \sin^{-1}(\sin \theta) = \theta] \quad (1) \end{aligned}$$

16. Write the value of $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Delhi 2011

 Firstly, find the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$, then solve it.

$$\begin{aligned} \sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] &= \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right] \\ &[\because \sin^{-1}(-\theta) = -\sin^{-1} \theta] \end{aligned}$$


$$= \sin\left[\frac{\pi}{3} + \sin^{-1}\left(\sin \frac{\pi}{6}\right)\right] \left[\because \sin \frac{\pi}{6} = \frac{1}{2} \right]$$

$$\begin{aligned} &= \sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right] = \sin \frac{\pi}{2} = 1 \\ &[\because \sin^{-1}(\sin \theta) = \theta] \quad (1) \end{aligned}$$

NOTE Please be careful that we do not write $\sin^{-1}(-\sin \theta) = \theta$.

17. Write the value of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

HOTS; Delhi 2011

 Firstly, we check the given angle in principal value. If it is so, then use the identity $\tan^{-1}(\tan\theta) = \theta$.

We know that, principal value branch of $\tan^{-1}x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$$\begin{aligned} \therefore \text{Principal value of } \tan^{-1}\left(\tan\frac{3\pi}{4}\right) &= \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right] \\ &\left[\begin{array}{l} \because \frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \\ \text{so, write } \frac{3\pi}{4} \text{ as } \left(\pi - \frac{\pi}{4}\right) \end{array} \right] \\ &= \tan^{-1}\left(-\tan\frac{\pi}{4}\right) \quad [\because \tan(\pi - \theta) = -\tan\theta] \\ &= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \quad [\because (-\tan\theta) = \tan(-\theta)] \\ &= -\frac{\pi}{4} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \\ \therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) &= -\frac{\pi}{4} \quad (1) \end{aligned}$$

NOTE Please be careful, we do not write

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4}, \text{ because } \frac{3\pi}{4} \notin \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

18. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

HOTS; Delhi 2011, 2009; All India 2009

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

$$\begin{aligned} \therefore \cos^{-1}\left(\cos \frac{7\pi}{6}\right) &= \cos^{-1}\left[\cos\left(2\pi - \frac{5\pi}{6}\right)\right] \\ &\left[\begin{array}{l} \because \frac{7\pi}{6} \notin [0, \pi], \text{ so} \\ \text{write } \frac{7\pi}{6} = \left(2\pi - \frac{5\pi}{6}\right) \end{array} \right] \end{aligned}$$

$$\begin{aligned} &= \cos^{-1}\left[\cos \frac{5\pi}{6}\right] \left[\begin{array}{l} \because \cos(2\pi - \theta) = \cos \theta \\ \text{and } \frac{5\pi}{6} \in [0, \pi] \end{array} \right] \\ &= \frac{5\pi}{6} \end{aligned}$$

$$\therefore \cos^{-1}\left(\cos \frac{7\pi}{6}\right) = \frac{5\pi}{6} \quad (1)$$

19. What is the principal value of

$$\cos^{-1}\left(\cos \frac{2\pi}{3}\right) + \sin^{-1}\left(\sin \frac{2\pi}{3}\right)?$$

All India 2011, 2008, 2009C

We know that, the principal value branch of $\cos^{-1} x$ is $[0, \pi]$ and for $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned} \therefore \cos^{-1}\left(\cos \frac{2\pi}{3}\right) + \sin^{-1}\left(\sin \frac{2\pi}{3}\right) & \\ &= \frac{2\pi}{3} + \sin^{-1}\left[\sin\left(\pi - \frac{\pi}{3}\right)\right] \\ & \left[\because \frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \text{ so} \right. \\ & \left. \text{write } \frac{2\pi}{3} \text{ as } \left(\pi - \frac{\pi}{3}\right) \right] \\ &= \frac{2\pi}{3} + \sin^{-1}\left(\sin \frac{\pi}{3}\right) \quad [\because \sin(\pi - \theta) = \sin \theta] \\ &= \frac{2\pi}{3} + \frac{\pi}{3} = \frac{3\pi}{3} = \pi \quad [\because \sin^{-1}(\sin \theta) = \theta] \quad (1) \end{aligned}$$

20. What is the principal value of $\tan^{-1}(-1)$?

Foreign 2011, 2008C

We know that, the principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\begin{aligned} \therefore \tan^{-1}(-1) &= \tan^{-1}\left(-\tan \frac{\pi}{4}\right) \quad \left[\because \tan \frac{\pi}{4} = 1\right] \\ &= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] \quad [\because (-\tan \theta) = \tan(-\theta)] \\ &= -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \end{aligned}$$

$$\text{Hence, } \tan^{-1}(-1) = -\frac{\pi}{4} \quad (1)$$

21. Using the principal values, write the value of

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right).$$

HOTS; All India 2011C

We know that, the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned}\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) &= \sin^{-1}\left(-\sin\frac{\pi}{3}\right) \\ &= \sin^{-1}\left[\sin\left(-\frac{\pi}{3}\right)\right] \quad [\because \sin(-\theta) = -\sin\theta] \\ &= -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad [\because \sin^{-1}(\sin\theta) = \theta] \\ \text{Hence, } \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) &= -\frac{\pi}{3} \quad (1)\end{aligned}$$

22. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$.

Delhi 2011C

We know that, the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned}\therefore \sin^{-1}\left(-\frac{1}{2}\right) &= \sin^{-1}\left(-\sin\frac{\pi}{6}\right) \\ &= \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right] \quad [\because \sin(-\theta) = -\sin\theta] \\ &= -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad [\because \sin^{-1}(\sin\theta) = \theta] \\ \text{Hence, } \sin^{-1}\left(-\frac{1}{2}\right) &= -\frac{\pi}{6} \quad (1)\end{aligned}$$

23. Write the principal value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Delhi 2010

We know that, the principal value branch of

$$\sin^{-1} x \text{ is } \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

$$\begin{aligned} \therefore \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) &= \sin^{-1} \left(\sin \frac{\pi}{3} \right) \left[\because \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \right] \\ &= \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \\ &\quad [\because \sin^{-1} (\sin \theta) = \theta] \quad (1) \end{aligned}$$

24. What is the principal value of $\sec^{-1} (-2)$?

All India 2010

We know that, the principal value branch of \sec^{-1} is $[0, \pi] - \left\{ \frac{\pi}{2} \right\}$.

$$\therefore \sec^{-1} (-2) = \sec^{-1} \left(-\sec \frac{\pi}{3} \right)$$

$$\left[\text{here, } \sec^{-1} \left(-\sec \frac{\pi}{3} \right) \neq -\frac{\pi}{3} \right]$$

$$\text{since, } -\frac{\pi}{3} \notin [0, \pi] - \left\{ \frac{\pi}{2} \right\}$$

$$= \sec^{-1} \left[\sec \left(\pi - \frac{\pi}{3} \right) \right] \left[\because \sec (\pi - \theta) = -\sec \theta \right]$$

$$= \sec^{-1} \left(\sec \frac{2\pi}{3} \right) = \frac{2\pi}{3} \in [0, \pi] - \left\{ \frac{\pi}{2} \right\}$$

$$[\because \sec^{-1} (\sec \theta) = \theta]$$

$$\text{Hence, } \sec^{-1} (-2) = \frac{2\pi}{3} \quad (1)$$

25. What is the domain of the function $\sin^{-1} x$?

Foreign 2010

$$\text{The domain of } \sin^{-1} x \text{ is } -1 \leq x \leq 1. \quad (1)$$

26. Using the principal values, find the value of

$$\cos^{-1} \left(\cos \frac{13\pi}{6} \right).$$

All India 2010C

As the principal value branch of $\cos^{-1} x$ is $[0, \pi]$.

$$\therefore \cos^{-1} \left(\cos \frac{13\pi}{6} \right) \neq \frac{13\pi}{6} \text{ as } \frac{13\pi}{6} \notin [0, \pi]$$

$$\begin{aligned} \text{Now, } \cos^{-1} \left(\cos \frac{13\pi}{6} \right) &= \cos^{-1} \left[\cos \left(2\pi + \frac{\pi}{6} \right) \right] \\ &= \cos^{-1} \left(\cos \frac{\pi}{6} \right) \quad [\because \cos(2\pi + \theta) = \cos \theta] \\ &= \frac{\pi}{6} \in [0, \pi] \quad [\because \cos^{-1}(\cos \theta) = \theta] \end{aligned}$$

$$\text{Hence, } \cos^{-1} \left(\cos \frac{13\pi}{6} \right) = \frac{\pi}{6} \quad (1)$$

27. If $\tan^{-1}(\sqrt{3}) + \cot^{-1} x = \frac{\pi}{2}$, then find the

value of x .

All India 2010C

$$\text{Given, } \tan^{-1} \sqrt{3} + \cot^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} \sqrt{3} = \frac{\pi}{2} - \cot^{-1} x$$

$$\Rightarrow \tan^{-1} \sqrt{3} = \tan^{-1} x \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right]$$

On equating both sides, we get

$$x = \sqrt{3} \quad (1)$$

28. Write the principal value of $\sin^{-1} \left(\sin \frac{3\pi}{5} \right)$.

Delhi 2009

As we know that, the principal value branch of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1} \left[\sin \left(\frac{3\pi}{5} \right) \right] \neq \frac{3\pi}{5} \left[\because \frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right]$$

$$\begin{aligned} \text{Now, } \sin^{-1} \left(\sin \frac{3\pi}{5} \right) &= \sin^{-1} \left[\sin \left(\pi - \frac{2\pi}{5} \right) \right] \\ &= \sin^{-1} \left(\sin \frac{2\pi}{5} \right) \left[\because \sin(\pi - \theta) = \sin \theta \right] \\ &= \frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \quad \left[\because \sin^{-1}(\sin \theta) = \theta \right] \end{aligned}$$

$$\text{Hence, } \sin^{-1} \left(\sin \frac{3\pi}{5} \right) = \frac{2\pi}{5} \quad (1)$$

29. Using the principal values, evaluate

$$\tan^{-1}(1) + \sin^{-1} \left(-\frac{1}{2} \right).$$

Delhi 2009C

We know that, the principal value of $\tan^{-1} x$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and that of \sin^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\begin{aligned} \therefore \tan^{-1}(1) + \sin^{-1}\left(-\frac{1}{2}\right) &= \tan^{-1}\left(\tan \frac{\pi}{4}\right) + \sin^{-1}\left(-\sin \frac{\pi}{6}\right) \\ &\quad \left[\because \tan \frac{\pi}{4} = 1 \text{ and } \sin \frac{\pi}{6} = \frac{1}{2}\right] \\ &= \tan^{-1}\left(\tan \frac{\pi}{4}\right) + \sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right] \\ &\quad \left[\because \sin(-\theta) = -\sin \theta\right] \\ &= \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12} \\ &\quad \left[\because \tan^{-1}(\tan \theta) = \theta \text{ and } \sin^{-1}(\sin \theta) = \theta\right] \quad (1) \end{aligned}$$

30. Find the principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.
All India 2008C

$$\cos^{-1} \frac{\sqrt{3}}{2} = \cos^{-1}\left(\cos \frac{\pi}{6}\right) \quad \left[\because \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}\right]$$

Since, $\frac{\pi}{6} \in [0, \pi]$

{As principal value branch of $\cos^{-1} x$ is $[0, \pi]$ }

\therefore

$$\cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6} \quad (1)$$

6 Mark Questions

31. Solve the following equation:

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$$

Foreign 2014; All India 2013; HOTS

$$\text{Given, } \cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$$

$$\Rightarrow \sin\left\{\frac{\pi}{2} - \tan^{-1} x\right\} = \sin\left(\cot^{-1} \frac{3}{4}\right)$$
$$\left[\because \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta\right] \quad (1)$$

On equating both sides, we get

$$\frac{\pi}{2} - \tan^{-1} x = \cot^{-1} \frac{3}{4} \quad (1)$$

$$\Rightarrow \tan^{-1} x + \cot^{-1} \frac{3}{4} = \frac{\pi}{2} \quad (1)$$

It is only possible, when $x = \frac{3}{4}$.

$$\left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R\right] \quad (1)$$

32. Solve following equation for x .

$$\tan^{-1} \left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x$$

Foreign 2011C, 08C; All India 2014C, 2010, 2009C

💡 Multiply both sides by 2 and then in LHS, use the relation $2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$ and solve it. Then, equate both sides to get the value of x .

$$\text{Given, } \tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \tan^{-1} x, x > 0$$

$$\Rightarrow 2 \tan^{-1} \left(\frac{1-x}{1+x} \right) = \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \left[\frac{2 \left(\frac{1-x}{1+x} \right)}{1 - \left(\frac{1-x}{1+x} \right)^2} \right] = \tan^{-1} x \quad (1\frac{1}{2})$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right]$$

$$\Rightarrow \tan^{-1} \left[\frac{2(1-x)(1+x)}{(1+x)^2 - (1-x)^2} \right] = \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \left(\frac{2-2x^2}{4x} \right) = \tan^{-1} x$$

$$[\because a^2 - b^2 = (a-b)(a+b)]$$

$$\Rightarrow \frac{1-x^2}{2x} = x \Rightarrow 1-x^2 = 2x^2$$

$$\Rightarrow 3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm \frac{1}{\sqrt{3}} \quad (1\frac{1}{2})$$

But given, $x > 0$

$$\therefore x = \frac{1}{\sqrt{3}} \quad (1)$$

33. Solve for x , $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$.

Delhi 2014C; All India 2009

Given equation is

$$2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2 \cos x}{1 - \cos^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \right] \quad (1\frac{1}{2})$$

$$\Rightarrow \frac{2 \cos x}{\sin^2 x} = \frac{2}{\sin x} \quad [\because \sin^2 x = 1 - \cos^2 x] \dots (i)$$

$$\Rightarrow \sin x \cdot \cos x - \sin^2 x = 0$$

$$\Rightarrow \sin x (\cos x - \sin x) = 0$$

$$\Rightarrow \sin x = 0 \text{ or } \cos x = \sin x$$

$$\Rightarrow \sin x = \sin 0 \text{ or } \cot x = 1 = \cot \pi/4$$

$$\Rightarrow x = 0 \text{ or } \frac{\pi}{4} \quad (1\frac{1}{2})$$

But here at $x = 0$, the given equation does not exist. Hence, $x = \frac{\pi}{4}$ is the only solution. (1)

34. Solve for x , $\tan^{-1} x + 2 \cot^{-1} x = \frac{2\pi}{3}$.

All India 2014C; Delhi 2009C

The given equation is $\tan^{-1} x + 2 \cot^{-1} x = \frac{2\pi}{3}$

We know that, $\cot^{-1} x = \tan^{-1} \frac{1}{x}$

So, the given equation can be written as

$$\tan^{-1} x + 2 \tan^{-1}\left(\frac{1}{x}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1} x + \tan^{-1}\left(\frac{2 \times \frac{1}{x}}{1 - \frac{1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\begin{aligned} & \left(\frac{1-x^2}{x^2} \right) \\ & \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right] \\ \Rightarrow & \tan^{-1} x + \tan^{-1} \left(\frac{\frac{2}{x}}{\frac{x^2-1}{x^2}} \right) = \frac{2\pi}{3} \\ \Rightarrow & \tan^{-1} x + \tan^{-1} \left(\frac{2x}{x^2-1} \right) = \frac{2\pi}{3} \quad (1\frac{1}{2}) \\ \Rightarrow & \tan^{-1} \left(\frac{x + \frac{2x}{x^2-1}}{1 - \frac{2x^2}{x^2-1}} \right) = \frac{2\pi}{3} \\ & \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \\ \Rightarrow & \frac{x^3 - x + 2x}{x^2 - 1 - 2x^2} = \tan \frac{2\pi}{3} \quad (1\frac{1}{2}) \\ \Rightarrow & \frac{x^3 + x}{-1 - x^2} = \tan \left(\pi - \frac{\pi}{3} \right) \\ \Rightarrow & \frac{x^3 + x}{-(1+x^2)} = -\tan \frac{\pi}{3} \\ & [\because \tan(\pi - \theta) = -\tan \theta] \\ \Rightarrow & \frac{x(1+x^2)}{-(1+x^2)} = -\sqrt{3} \Rightarrow x = \sqrt{3} \quad (1) \end{aligned}$$

35. Prove that

$$\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}; x \in \left(0, \frac{\pi}{4} \right)$$

Delhi 2014C, 2011; All India 2009

$$\begin{aligned}
 \cdot \text{LHS} &= \cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right] \\
 &= \cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right. \\
 &\quad \left. \times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right] \quad (1)
 \end{aligned}$$

[multiplying and dividing by conjugate of denominator, i.e. by $\sqrt{1+\sin x} + \sqrt{1-\sin x}$]

$$= \cot^{-1} \left[\frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2} \right] \quad (1)$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$= \cot^{-1} \left[\frac{1 + \sin x + 1 - \sin x + 2\sqrt{1 - \sin^2 x}}{1 + \sin x - 1 + \sin x} \right]$$

$$= \cot^{-1} \left[\frac{2 + 2 \cos x}{2 \sin x} \right] \quad [\because 1 - \sin^2 x = \cos^2 x] \quad (1)$$

$$= \cot^{-1} \left[\frac{1 + \cos x}{\sin x} \right] = \cot^{-1} \left[\frac{2 \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \right]$$

$$\left[\because \cos x = 2 \cos^2 \frac{x}{2} - 1 \text{ and } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \right]$$

$$= \cot^{-1} \left[\cot \frac{x}{2} \right] = \frac{x}{2} = \text{RHS} \quad (1)$$

Hence proved.

36. Prove that

$$2 \tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2 \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}.$$

Delhi 2014

$$\therefore \text{LHS} = 2 \tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2 \tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2 \left\{ \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} \right\} + \sec^{-1} \frac{5\sqrt{2}}{7}$$

$$= 2 \tan^{-1} \left\{ \frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{5} \times \frac{1}{8}} \right\} + \tan^{-1} \sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1}$$

$$\left[\begin{array}{l} \because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left\{ \frac{x+y}{1-xy} \right\} \quad (1) \\ \text{and } \sec^{-1} x = \tan^{-1} \sqrt{x^2 - 1} \end{array} \right]$$

$$= 2 \tan^{-1} \frac{13}{39} + \tan^{-1} \sqrt{\frac{50}{49} - 1}$$

$$= 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} \quad (2)$$

$$= \tan^{-1} \left\{ \frac{2 \times \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} \right\} + \tan^{-1} \frac{1}{7}$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left\{ \frac{2x}{1-x^2} \right\} \right] \quad (1)$$

$$= \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{1}{7} = \tan^{-1} \left\{ \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}} \right\}$$

$$= \tan^{-1}(1) = \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4} = \text{RHS} \quad (1)$$

$[\because \tan^{-1}(\tan \theta) = \theta]$ Hence proved.

37. Prove that $\tan^{-1} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right]$

$$= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, \quad \frac{-1}{\sqrt{2}} \leq x \leq 1.$$

All India 2014, 2011, 2014C; HOTS

$$\text{LHS} = \tan^{-1} \left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right]$$

Let $x = \cos \theta$, so that $\cos^{-1} x = \theta$ (1)

$$\begin{aligned} \text{Then, LHS} &= \tan^{-1} \left[\frac{\sqrt{1+\cos \theta} - \sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta} + \sqrt{1-\cos \theta}} \right] \\ &= \tan^{-1} \left[\frac{\sqrt{2} \cos \frac{\theta}{2} - \sqrt{2} \sin \frac{\theta}{2}}{\sqrt{2} \cos \frac{\theta}{2} + \sqrt{2} \sin \frac{\theta}{2}} \right] \end{aligned}$$

$$\left[\because 1 + \cos \theta = 2 \cos^2 \frac{\theta}{2} \text{ and } 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \right]$$

(1)

$$= \tan^{-1} \left[\frac{\cos \frac{\theta}{2} - \sin \frac{\theta}{2}}{\cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \right] = \tan^{-1} \left[\frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} \right]$$

[dividing numerator and denominator by $\cos(\theta/2)$] (1)

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \right]$$

$$\left[\because \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \right]$$

$$= \frac{\pi}{4} - \frac{\theta}{2}$$

$$[\because \tan^{-1}(\tan \theta) = \theta]$$

$$= \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x$$

$$[\because \theta = \cos^{-1} x]$$

= RHS

Hence proved.

38. If $\tan^{-1} \left(\frac{x-2}{x-4} \right) + \tan^{-1} \left(\frac{x+2}{x+4} \right) = \frac{\pi}{4}$, then

find the value of x .

All India 2014

$$\text{Given, } \tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x-2}{x-4} + \frac{x+2}{x+4}}{1 - \left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right] = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right] \quad (1)$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{(x-2)(x+4) + (x+2)(x-4)}{(x-4)(x+4)}}{\frac{(x-4)(x+4) - (x-2)(x+2)}{(x-4)(x+4)}}\right] = \frac{\pi}{4}$$

$$\Rightarrow \frac{x^2 - 2x + 4x - 8 + x^2 - 4x + 2x - 8}{(x^2 - 16) - (x^2 - 4)}$$

$$= \tan \frac{\pi}{4} \quad (1)$$

$$\Rightarrow \frac{2x^2 - 16}{-12} = 1$$

$$\Rightarrow 2x^2 - 16 = -12$$

$$\Rightarrow 2x^2 = -12 + 16$$

$$\Rightarrow 2x^2 = 4 \Rightarrow x^2 = 2$$

$$\therefore x = \pm \sqrt{2} \quad (1)$$

Hence, $\sqrt{2}$ and $-\sqrt{2}$ are the required values of x .

(1)

39. Prove that

$$\cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\} = \frac{\pi}{3}$$

All India 2014C

We have to prove

$$\cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\} = \frac{\pi}{3}$$

$$\text{LHS} = \cos^{-1}(x) + \cos^{-1}\left\{\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right\}$$

$$\text{Let } \cos^{-1} x = \alpha \Rightarrow x = \cos \alpha \quad (1)$$

$$\text{Then, } \text{LHS} = \alpha + \cos^{-1}$$

$$\left[\cos \alpha \cdot \cos \frac{\pi}{3} + \frac{\sqrt{3}}{2} \sqrt{1 - \cos^2 \alpha} \right]$$

$$= \alpha + \cos^{-1} \left[\cos \frac{\pi}{3} \cos \alpha + \sin \frac{\pi}{3} \sin \alpha \right]$$

$$\left[\because \sin \alpha = \sqrt{1 - \cos^2 \alpha}, \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \right] (1)$$

$$= \alpha + \cos^{-1} \left[\cos \left(\frac{\pi}{3} - \alpha \right) \right]$$

$$[\because \cos(A - B) = \cos A \cos B + \sin A \sin B] (1)$$

$$= \alpha + \frac{\pi}{3} - \alpha = \frac{\pi}{3} = \text{RHS} \quad (1)$$

40. Prove that $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$

Foreign 2014

To prove, $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$

$$\text{LHS} = \cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$$

$$= \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18}$$

$$\left[\because \cot^{-1} x = \frac{1}{\tan^{-1} x} \right] \quad (1)$$

$$= \tan^{-1} \left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \left(\frac{1}{8}\right)} \right) + \tan^{-1} \frac{1}{18}$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1} \left(\frac{A+B}{1-AB} \right) \right] \quad (1)$$

$$= \tan^{-1} \left(\frac{15}{55} \right) + \tan^{-1} \frac{1}{18}$$

$$= \tan^{-1} \left(\frac{3}{11} \right) + \tan^{-1} \frac{1}{18}$$

$$= \tan^{-1} \left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \left(\frac{3}{11}\right) \times \left(\frac{1}{18}\right)} \right) = \tan^{-1} \left(\frac{65}{195} \right) \quad (1)$$

$$= \tan^{-1} \left(\frac{1}{3} \right) = \cot^{-1} 3 = \text{RHS} \quad (1)$$

Hence proved.

41. Prove the following:

$$\cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right] = \frac{x}{2}; x \in \left(0, \frac{\pi}{4} \right)$$

Delhi 2014, 2011; HOTS; All India 2009



Use the relation

$$1 + \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2$$

For $x \in \left(0, \frac{\pi}{4} \right)$,

$$1 - \sin x = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= \left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2 \text{ and then simplify.}$$

$$\text{LHS} = \cot^{-1} \left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right)$$

$$= \cot^{-1} \left[\frac{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} + \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2}}{\sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} - \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2}} \right]$$

(1½)

$$\text{LHS} = \cot^{-1} \left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2} + \cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2} - \cos \frac{x}{2} + \sin \frac{x}{2}} \right) \quad (1)$$

$$= \cot^{-1} \left(\frac{2 \cos \frac{x}{2}}{2 \sin \frac{x}{2}} \right) = \cot^{-1} \left(\cot \frac{x}{2} \right) \quad (1/2)$$

\therefore The principal value branch of $\cot^{-1} x$ is $(0, \pi)$.

$$\therefore \cot^{-1} \left(\cot \frac{x}{2} \right) = \frac{x}{2}$$

∴ LHS = RHS

Hence proved.

NOTE If $x \in \left(0, \frac{\pi}{2}\right)$, then $\sqrt{1 - \sin x} = \cos \frac{x}{2} - \sin \frac{x}{2}$

and if $x \in \left(\frac{\pi}{2}, \pi\right)$, then $\sqrt{1 - \sin x} = \sin \frac{x}{2} - \cos \frac{x}{2}$

Alternate Method

$$\begin{aligned} \text{LHS} &= \cot^{-1} \left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right) \\ &= \cot^{-1} \left(\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}} \right) \\ &\quad \times \frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}} \end{aligned}$$

[by rationalising denominator] (1)

$$= \cot^{-1} \left[\frac{(\sqrt{1 + \sin x} + \sqrt{1 - \sin x})^2}{(\sqrt{1 + \sin x})^2 - (\sqrt{1 - \sin x})^2} \right]$$

$$[\because (a + b)(a - b) = a^2 - b^2]$$

$$= \cot^{-1} \left(\frac{1 + \sin x + 1 - \sin x + 2\sqrt{1 - \sin^2 x}}{1 + \sin x - 1 + \sin x} \right)$$

$$[\because (a + b)^2 = a^2 + b^2 + 2ab]$$

$$= \cot^{-1}\left(\frac{2 + 2 \cos x}{2 \sin x}\right) [\because \cos x = \sqrt{1 - \sin^2 x}] \quad (1)$$

$$= \cot^{-1}\left(\frac{1 + \cos x}{\sin x}\right) = \cot^{-1}\left(\frac{2 \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}\right) \quad (1)$$

$$\left[\because 1 + \cos \theta = 2 \cos^2 \frac{\theta}{2} \text{ and } \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right]$$

$$= \cot^{-1}\left[\cot \frac{x}{2}\right] = \frac{x}{2} = \text{RHS}$$

$$[\because \cot^{-1}(\cot \theta) = \theta] \quad (1)$$

42. Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

All India 2014C; Delhi 2012, 2010C

To prove, $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$

Let $\sin^{-1}\left(\frac{8}{17}\right) = x$... (i)

and $\sin^{-1}\left(\frac{3}{5}\right) = y$... (ii)

$\Rightarrow \sin x = \frac{8}{17}$ and $\sin y = \frac{3}{5}$ (1)

Now, $\cos^2 x = 1 - \sin^2 x$
 $= 1 - \frac{64}{289} = \frac{225}{289} \Rightarrow \cos x = \sqrt{\frac{225}{289}}$
 $\Rightarrow \cos x = \frac{15}{17}$ (1)

Also, $\cos^2 y = 1 - \sin^2 y = 1 - \frac{9}{25}$
 $\Rightarrow \cos y = \sqrt{\frac{16}{25}} \Rightarrow \cos y = \frac{4}{5}$ (1)

We know that,

$\cos(x + y) = \cos x \cos y - \sin x \sin y$
 $\Rightarrow \cos(x + y) = \left(\frac{15}{17} \times \frac{4}{5}\right) - \left(\frac{8}{17} \times \frac{3}{5}\right)$
 $\Rightarrow \cos(x + y) = \frac{60}{85} - \frac{24}{85} = \frac{36}{85}$
 $\Rightarrow x + y = \cos^{-1}\left(\frac{36}{85}\right)$... (iii)
 $\Rightarrow \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \cos^{-1}\frac{36}{85}$ (1)

[from Eqs. (i), (ii) and (iii)] **Hence proved.**

43. Show that $\tan\left(\frac{1}{2} \sin^{-1} \frac{3}{4}\right) = \frac{4 - \sqrt{7}}{3}$ All India 2013

To prove, $\tan\left(\frac{1}{2} \sin^{-1} \frac{3}{4}\right) = \frac{4 - \sqrt{7}}{3}$

LHS = $\tan\left(\frac{1}{2} \sin^{-1} \frac{3}{4}\right)$... (i)

1 ... (3) ...

$$\text{Let } \frac{1}{2} \sin^{-1} \left(\frac{3}{4} \right) = \theta \quad \dots \text{(ii)}$$

$$\Rightarrow \sin^{-1} \left(\frac{3}{4} \right) = 2\theta \Rightarrow \sin 2\theta = \frac{3}{4} \quad \text{(1)}$$

$$\Rightarrow \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{3}{4} \left[\because \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \right]$$

$$\Rightarrow 8 \tan \theta = 3 + 3 \tan^2 \theta$$

$$\Rightarrow 3 \tan^2 \theta - 8 \tan \theta + 3 = 0$$

Now, by Sridharacharya's rule

$$\tan \theta = \frac{8 \pm \sqrt{64 - 36}}{2 \times 3} = \frac{8 \pm \sqrt{28}}{6} \quad \text{(1)}$$

$$\Rightarrow \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3} \Rightarrow \theta = \tan^{-1} \left(\frac{4 \pm \sqrt{7}}{3} \right)$$

$$[\because \tan \theta = \phi \Rightarrow \theta = \tan^{-1} \phi]$$

So, from Eq. (ii), we get

$$\frac{1}{2} \sin^{-1} \left(\frac{3}{4} \right) = \tan^{-1} \left(\frac{4 \pm \sqrt{7}}{3} \right) \quad \text{(1)}$$

Taking (-)ve sign, we get

$$\frac{1}{2} \sin^{-1} \left(\frac{3}{4} \right) = \tan^{-1} \left(\frac{4 - \sqrt{7}}{3} \right)$$

On taking tan both sides, we get

$$\tan \left(\frac{1}{2} \sin^{-1} \frac{3}{4} \right) = \tan \left\{ \tan^{-1} \left(\frac{4 - \sqrt{7}}{3} \right) \right\}$$

$$\Rightarrow \tan \left(\frac{1}{2} \sin^{-1} \frac{3}{4} \right) = \frac{4 - \sqrt{7}}{3}$$

$$[\because \tan (\tan^{-1} \theta) = \theta] \text{ (1)}$$

= RHS

Hence proved.

44. Prove that $\sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$.

Delhi 2013C

$$\text{To prove, } \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}$$

$$\text{Let } \sin^{-1} \frac{8}{17} = x \text{ and } \sin^{-1} \frac{3}{5} = y$$

$$\Rightarrow \sin x = \frac{8}{17} \text{ and } \sin y = \frac{3}{5} \quad (1)$$

$$\text{Now, } \cos^2 x = 1 - \sin^2 x = 1 - \frac{64}{289} = \frac{225}{289}$$

$$\Rightarrow \cos x = \sqrt{\frac{225}{289}} = \frac{15}{17}$$

$$\text{Also, } \cos^2 y = 1 - \sin^2 y = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow \cos y = \sqrt{16/25} = 4/5 \quad (1)$$

We know that,

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ &= \frac{15}{17} \times \frac{4}{5} - \frac{8}{17} \times \frac{3}{5} = \frac{60}{85} - \frac{24}{85} = \frac{36}{85} \end{aligned}$$

$$\Rightarrow x+y = \cos^{-1} \left(\frac{36}{85} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{8}{17} \right) + \sin^{-1} \left(\frac{3}{5} \right) = \cos^{-1} \left(\frac{36}{85} \right) \quad \dots(i)$$

$$\text{Again, let } \cos^{-1} \left(\frac{36}{85} \right) = z \Rightarrow \cos z = \frac{36}{85}$$

$$\text{then, } \tan z = \frac{77}{36} \quad (1)$$

$$\Rightarrow z = \tan^{-1} \frac{77}{36} \Rightarrow \cos^{-1} \frac{36}{85} = \tan^{-1} \frac{77}{36}$$

On putting this value in Eq. (i), we get

$$\sin^{-1} \left(\frac{8}{17} \right) + \sin^{-1} \left(\frac{3}{5} \right) = \tan^{-1} \left(\frac{77}{36} \right) \quad (1)$$

Hence proved.

45. Solve for x , $\tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$.

Delhi 2013C, 2009 ; All India 2009C, 2008

$$\text{Given equation is } \tan^{-1} 3x + \tan^{-1} 2x = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left(\frac{3x + 2x}{1 - 3x \times 2x} \right) = \frac{\pi}{4} \quad (1)$$

$$\left[\because \tan^{-1} A + \tan^{-1} B = \tan^{-1} \left(\frac{A + B}{1 - AB} \right) \right]$$

$$\Rightarrow \tan^{-1} \left(\frac{5x}{1 - 6x^2} \right) = \frac{\pi}{4} \Rightarrow \frac{5x}{1 - 6x^2} = \tan \frac{\pi}{4}$$

$$[\because \tan^{-1}(\theta) = \phi \Rightarrow \theta = \tan \phi]$$

$$\Rightarrow \frac{5x}{1 - 6x^2} = 1$$

$$\Rightarrow 5x = 1 - 6x^2 \quad (1)$$

$$\Rightarrow 6x^2 + 5x - 1 = 0$$

$$\Rightarrow 6x^2 + 6x - x - 1 = 0$$

$$\Rightarrow 6x(x + 1) - 1(x + 1) = 0 \quad (1)$$

$$\Rightarrow (6x - 1)(x + 1) = 0$$

$$\Rightarrow 6x - 1 = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow x = 1/6 \text{ or } x = -1 \quad (1)$$

But $x = -1$ does not satisfy the given equation.

Hence, required value of x is $\frac{1}{6}$.

46. Find the value of the following:

$$\tan \frac{1}{2} \left[\sin^{-1} \left(\frac{2x}{1 + x^2} \right) + \cos^{-1} \left(\frac{1 - y^2}{1 + y^2} \right) \right],$$

if $|x| < 1, y > 0$ and $xy < 1$

Delhi 2013



Firstly, use the relation

$$2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right) = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \text{ to}$$

convert into \tan^{-1} , then use identity relation

$$\tan(\tan^{-1} \theta) = \theta.$$

$$\begin{aligned} \tan \left\{ \frac{1}{2} \sin^{-1} \left(\frac{2x}{1+x^2} \right) + \frac{1}{2} \cos^{-1} \left(\frac{1-y^2}{1+y^2} \right) \right\} \\ = \tan \left[\frac{1}{2} (2 \tan^{-1} x) + \frac{1}{2} (2 \tan^{-1} y) \right] \quad (2) \end{aligned}$$

$$\left[\because 2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right) = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \right]$$

$$= \tan(\tan^{-1} x + \tan^{-1} y)$$

$$= \tan \left[\tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$= \frac{x+y}{1-xy} \quad [\because \tan(\tan^{-1} \theta) = \theta] \quad (2)$$

47. Prove that

$$\tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{5} \right) + \tan^{-1} \left(\frac{1}{8} \right) = \frac{\pi}{4}.$$

Delhi 2013; All India 2011, 2008C

To prove,

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$$

$$\begin{aligned} \text{LHS} &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) \\ &= \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{10}}\right) + \tan^{-1}\left(\frac{1}{8}\right) \quad (1\frac{1}{2}) \end{aligned}$$

$$[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1]$$

$$= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{72}}\right) \quad (1\frac{1}{2})$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1 \right]$$

$$= \tan^{-1}\left(\frac{56+9}{72-7}\right) = \tan^{-1}\left(\frac{65}{65}\right)$$


$$= \tan^{-1}(1) = \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS} \quad (1)$$

Hence proved.

48. Prove that

$$\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Delhi 2012;HOTS

 Firstly, use the relation $\cos\theta = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}$ and $\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$. After that use the relation $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ and simplify it.

$$\begin{aligned}
 \text{LHS} &= \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) \\
 &= \tan^{-1} \left(\frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} \right) \\
 &\quad \left[\begin{array}{l} \because \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \\ \text{and } \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \end{array} \right] \quad (1) \\
 &= \tan^{-1} \left[\frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} \right] \\
 &\quad [\because a^2 - b^2 = (a - b)(a + b) \text{ and } (a + b)^2 = a^2 + 2ab + b^2] \\
 &= \tan^{-1} \left(\frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} \right) \quad (1)
 \end{aligned}$$

On dividing the numerator and denominator by $\cos \frac{x}{2}$, we get

$$\text{LHS} = \tan^{-1} \left(\frac{\frac{\cos \frac{x}{2}}{\cos \frac{x}{2}} - \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}}{\frac{\cos \frac{x}{2}}{\cos \frac{x}{2}} + \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}} \right) = \tan^{-1} \left(\frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}} \right)$$

$$= \tan^{-1} \left(\frac{\tan \frac{\pi}{4} - \tan \frac{x}{2}}{1 + \tan \frac{\pi}{4} \cdot \tan \frac{x}{2}} \right) \quad (1)$$

$$\left[\begin{array}{l} \because 1 = \tan \frac{\pi}{4} \text{ and} \\ 1 \cdot \tan \frac{x}{2} = \tan \frac{\pi}{4} \cdot \tan \frac{x}{2} \end{array} \right]$$

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \frac{x}{2} \right) \right]$$

$$\left[\because \frac{\tan A - \tan B}{1 + \tan A \tan B} = \tan(A - B) \right]$$

$$= \frac{\pi}{4} - \frac{x}{2}$$

$$\left[\because \tan^{-1}(\tan \theta) = \theta; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right]$$

= RHS

(1) Hence proved.

49. Prove that

$$\cos^{-1} \left(\frac{4}{5} \right) + \cos^{-1} \left(\frac{12}{13} \right) = \cos^{-1} \left(\frac{33}{65} \right).$$

All India 2012: Delhi 2010C, 2009

$$\text{To prove, } \cos^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \cos^{-1}\left(\frac{33}{65}\right)$$

$$\text{Let } \cos^{-1}\left(\frac{4}{5}\right) = x \quad \dots(i)$$

$$\text{and } \cos^{-1}\left(\frac{12}{13}\right) = y \quad \dots(ii)$$

$$\Rightarrow \cos x = \frac{4}{5} \text{ and } \cos y = \frac{12}{13} \quad (1)$$

We know that,

$$\sin^2 x = 1 - \cos^2 x = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\Rightarrow \sin x = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\text{and } \sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$$

$$\sin y = \sqrt{\frac{25}{169}} \Rightarrow \frac{5}{13} \quad (1)$$

Now, we know that,

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\Rightarrow \cos(x + y) = \left(\frac{4}{5} \times \frac{12}{13}\right) - \left(\frac{3}{5} \times \frac{5}{13}\right)$$
$$= \frac{48}{65} - \frac{15}{65}$$

$$\Rightarrow \cos(x + y) = \frac{33}{65} \quad (1)$$

$$\Rightarrow x + y = \cos^{-1} \frac{33}{65}$$

$$\Rightarrow \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65} \quad \dots(iii)$$

[From Eqs. (i), (ii) and (iii)] (1) Hence proved.

50. Prove that $\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$.

Foreign 2012

To prove,

$$\sin^{-1}\left(\frac{63}{65}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

$$\text{RHS} = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right)$$

$$\text{Let } \sin^{-1}\frac{5}{13} = x \Rightarrow \sin x = \frac{5}{13} \quad \dots(i)$$

$$\text{and } \cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5} \quad \dots(ii) \text{ (1)}$$

$$\begin{aligned} \text{Also, } \cos x &= \sqrt{1 - \sin^2 x} \\ &= \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13} \end{aligned}$$

$$\begin{aligned} \text{and } \sin y &= \sqrt{1 - \cos^2 y} \\ &= \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad \text{(1)} \end{aligned}$$

We know that,

$$\begin{aligned} \sin(x + y) &= \sin x \cos y + \cos x \sin y \\ &= \frac{5}{13} \times \frac{3}{5} + \frac{12}{13} \times \frac{4}{5} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65} \quad \text{(1)} \end{aligned}$$

$$\Rightarrow x + y = \sin^{-1}\left(\frac{63}{65}\right) \quad \dots(iii)$$

$$[\because \sin \theta = \phi \Rightarrow \theta = \sin^{-1} \phi]$$

$$\Rightarrow \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{63}{65}\right) \quad \text{(1)}$$

[from Eqs. (i), (ii) and (iii)]

Hence proved.

51. Solve for x ,

$$2 \tan^{-1}(\sin x) = \tan^{-1}(2 \sec x), \quad x \neq \frac{\pi}{2}$$

Foreign 2012

To solve, $2 \tan^{-1}(\sin x) = \tan^{-1}(2 \sec x)$

$$\Rightarrow \tan^{-1}\left(\frac{2 \sin x}{1 - \sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\cos x}\right) \quad (1/2)$$

$$\left[\begin{array}{l} \because 2 \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \\ \phantom{2 \tan^{-1} x} = \tan^{-1}\left(\frac{1}{\cos x}\right) \end{array} \right]$$

On comparing, we get


$$\Rightarrow \frac{2 \sin x}{\cos^2 x} = \frac{2}{\cos x} \quad (1)$$

$$\Rightarrow \tan x = 1 \quad (1/2)$$

$$\Rightarrow x = \tan^{-1}(1) = \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4} \quad (1)$$

52. Find the value of $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$.

Delhi 2011

 Use the relation

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x - y}{1 + xy} \right)$$

We have, $\tan^{-1} \left(\frac{x}{y} \right) - \tan^{-1} \left(\frac{x - y}{x + y} \right)$

$$= \tan^{-1} \left(\frac{\frac{x}{y} - \frac{x - y}{x + y}}{1 + \frac{x}{y} \cdot \frac{x - y}{x + y}} \right) \quad (1\frac{1}{2})$$

$$\left[\because \tan^{-1} \theta - \tan^{-1} \phi = \tan^{-1} \left(\frac{\theta - \phi}{1 + \theta \cdot \phi} \right) \right]$$

$$= \tan^{-1} \left[\frac{x(x + y) - y(x - y)}{y(x + y) + x(x - y)} \right]$$

$$= \tan^{-1} \left(\frac{x^2 + xy - xy + y^2}{xy + y^2 + x^2 - xy} \right) \quad (1\frac{1}{2})$$

$$= \tan^{-1} \left(\frac{x^2 + y^2}{y^2 + x^2} \right) = \tan^{-1}(1)$$

$$\left[\because \text{principal value branch of } \tan^{-1} x \text{ is } \left(\frac{-\pi}{2}, \frac{\pi}{2} \right) \right]$$

$$= \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4} \quad (1)$$

53. Prove that

$$2 \tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{7} \right) = \tan^{-1} \left(\frac{31}{17} \right)$$

All India 2011; Delhi 2009C, 2008C

💡 Use the relation, $2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$
 and then $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$

$$\begin{aligned} \text{LHS} &= 2 \tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{7} \right) \\ &= \tan^{-1} \left[\frac{2 \times (1/2)}{1 - (1/2)^2} \right] + \tan^{-1} \frac{1}{7} \\ &\quad \left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right] \quad (1\frac{1}{2}) \end{aligned}$$

$$\begin{aligned} &= \tan^{-1} \left(\frac{1}{1 - \frac{1}{4}} \right) + \tan^{-1} \left(\frac{1}{7} \right) \\ &= \tan^{-1} \left(\frac{1}{3/4} \right) + \tan^{-1} \left(\frac{1}{7} \right) \\ &= \tan^{-1} \left(\frac{4}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right) = \tan^{-1} \left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \frac{4}{3} \times \frac{1}{7}} \right) \quad (1\frac{1}{2}) \\ &\quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \\ &= \tan^{-1} \left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}} \right) = \tan^{-1} \frac{31}{17} = \text{RHS} \quad (1) \end{aligned}$$

Hence proved.

54. Prove that $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \left(\frac{1}{3} \right) = \frac{9}{4} \sin^{-1} \left(\frac{2\sqrt{2}}{3} \right)$.

Foreign 2011

Method 1

$$\text{Let } \sin^{-1}\left(\frac{1}{3}\right) = x \text{ and } \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$$

Then, we get

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}$$

$$\text{Now, } \cos^2 x = 1 - \sin^2 x = 1 - \frac{1}{9} = \frac{8}{9}$$

$$\Rightarrow \cos x = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \quad (1)$$

$$\text{Similarly, } \cos^2 y = 1 - \sin^2 y = 1 - \frac{8}{9} = \frac{1}{9}$$

$$\Rightarrow \cos y = \sqrt{\frac{1}{9}} = \frac{1}{3} \quad (1)$$

$$\text{Now, } \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{1}{3} \times \frac{1}{3} + \frac{2\sqrt{2}}{3} \times \frac{2\sqrt{2}}{3}$$

$$= \frac{1}{9} + \frac{8}{9} = \frac{9}{9} = 1 \quad (1)$$

$$\Rightarrow \sin(x + y) = 1 = \sin \frac{\pi}{2}$$

$\left[\because \text{principal value branch of } \sin^{-1} x \text{ is} \right.$

$$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\begin{aligned}
 x + y &= \frac{\pi}{2} \\
 \Rightarrow \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) &= \frac{\pi}{2} \\
 \left[\because x = \sin^{-1}\left(\frac{1}{3}\right) \text{ and } y = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \right] \\
 \Rightarrow \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) &= \frac{9\pi}{8} \\
 & \text{[multiplying both sides by } 9/4] \\
 \Rightarrow \frac{9\pi}{8} - \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right) &= \frac{9}{4} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \quad (1)
 \end{aligned}$$

Hence proved.

Method II

To prove that

$$\begin{aligned}
 \frac{9\pi}{8} - \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right) &= \frac{9}{4} \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) \\
 \text{LHS} &= \frac{9\pi}{8} - \frac{9}{4} \sin^{-1}\left(\frac{1}{3}\right) \\
 &= \frac{9}{4} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{3}\right) \right] \quad (1) \\
 &= \frac{9}{4} \left[\cos^{-1}\left(\frac{1}{3}\right) \right] \left[\because \cos^{-1} \theta = \frac{\pi}{2} - \sin^{-1} \theta \right] \quad (1) \\
 &= \frac{9}{4} \sin^{-1}\left(\sqrt{1 - \frac{1}{9}}\right) \\
 & \quad \left[\because \cos^{-1} x = \sin^{-1} \sqrt{1 - x^2} \right] \quad (1) \\
 &= \frac{9}{4} \sin^{-1}\left(\sqrt{\frac{8}{9}}\right) \quad (1) \\
 &= \text{RHS} \quad \text{Hence proved.}
 \end{aligned}$$

55. Prove that $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \frac{1}{2} \tan^{-1} \frac{4}{3}$.
All India 2011C

To prove,

$$\tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right) = \left(\frac{1}{2} \right) \tan^{-1} \left(\frac{4}{3} \right) \quad \dots(i)$$

Above equation may be rewritten as

$$2 \left[\tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right) \right] = \tan^{-1} \left(\frac{4}{3} \right) \quad \dots(ii)$$

(1/2)

$$\text{Now, LHS} = 2 \left[\tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right) \right]$$

$$= 2 \left[\tan^{-1} \left(\frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4} \times \frac{2}{9}} \right) \right] \quad (1)$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$= 2 \tan^{-1} \left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}} \right) = 2 \tan^{-1} \left(\frac{17}{34} \right)$$

$$= 2 \tan^{-1} \left(\frac{1}{2} \right) = \tan^{-1} \left[\frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2} \right)^2} \right] \quad (1)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right]$$

$$= \tan^{-1} \left(\frac{1}{1 - \frac{1}{4}} \right) = \tan^{-1} \left(\frac{4}{3} \right) = \text{RHS} \quad (1\frac{1}{2})$$

Hence proved.

56. Solve for x , $\cos (2 \sin^{-1} x) = \frac{1}{9}, x > 0$.

All India 2011C; HOTS

Given equation is

$$\cos(2 \sin^{-1} x) = \frac{1}{9}, x > 0 \quad \dots(i)$$

$$\text{Put } \sin^{-1} x = y \Rightarrow x = \sin y \quad (1/2)$$

$$\text{Then, Eq. (i) becomes, } \cos 2y = \frac{1}{9}$$

$$[\because \sin y = x]$$

$$\Rightarrow 1 - 2 \sin^2 y = \frac{1}{9}$$

$$[\because \cos 2\theta = 1 - 2 \sin^2 \theta] \quad (1)$$

$$\Rightarrow 2 \sin^2 y = 1 - \frac{1}{9} = \frac{8}{9} \quad (1/2)$$

$$\Rightarrow \sin^2 y = \frac{4}{9} \Rightarrow x^2 = \frac{4}{9} \quad [\because \sin y = x]$$

$$\Rightarrow x = \pm \frac{2}{3} \quad [\text{taking square root}] \quad (1)$$

But it is given that, $x > 0$

$$\therefore x = \frac{2}{3} \quad (1)$$

Alternate Method

$$\text{Given equation is } \cos(2 \sin^{-1} x) = \frac{1}{9}, x > 0$$

$$\Rightarrow \cos(\sin^{-1} 2x \sqrt{1-x^2}) = \frac{1}{9}$$

$$[\because 2 \sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})] \quad (1)$$

$$\Rightarrow \cos \left[\cos^{-1} \sqrt{1 - (2x\sqrt{1-x^2})^2} \right] = \frac{1}{9}$$

$$[\because \sin^{-1} x = \cos^{-1} \sqrt{1-x^2}] \quad (1)$$

$$\Rightarrow \sqrt{1 - 4x^2(1-x^2)} = \frac{1}{9} \quad [\because \cos(\cos^{-1} \theta) = \theta]$$

On squaring both sides, we get

$$81(1 - 4x^2 + 4x^4) = 1$$

$$\Rightarrow 324x^4 - 324x^2 + 80 = 0$$

$$\Rightarrow 81x^4 - 81x^2 + 20 = 0$$

[dividing both sides by 4]

$$\Rightarrow 81x^4 - 45x^2 - 36x^2 + 20 = 0$$

$$\Rightarrow 9x^2(9x^2 - 5) - 4(9x^2 - 5) = 0$$

$$\Rightarrow (9x^2 - 5)(9x^2 - 4) = 0$$

$$\Rightarrow x^2 = \frac{5}{9} \text{ or } \frac{4}{9} \Rightarrow x = \pm \frac{\sqrt{5}}{3} \text{ or } \pm \frac{2}{3}$$

But $x > 0$

$$\therefore x = +\frac{\sqrt{5}}{3} \text{ or } \frac{2}{3} \quad (1)$$


But here, $x = \frac{\sqrt{5}}{3}$ does not satisfy the given equation.

$$\therefore x = 2/3 \text{ is the only solution.} \quad (1)$$

NOTE While solving an equation, please be careful on squaring the equation. Sometimes, it may give extra value, which do not satisfy the given equation.

57. Prove that $2 \tan^{-1} \frac{3}{4} - \tan^{-1} \frac{17}{31} = \frac{\pi}{4}$.

Delhi 2011C

 Firstly, apply $2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$ to evaluate $2 \tan^{-1} \left(\frac{3}{4} \right)$ and then apply $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right)$ and get the desired result.

To prove that $2 \tan^{-1} \left(\frac{3}{4} \right) - \tan^{-1} \left(\frac{17}{31} \right) = \frac{\pi}{4}$

$$\begin{aligned} \text{LHS} &= 2 \tan^{-1} \left(\frac{3}{4} \right) - \tan^{-1} \left(\frac{17}{31} \right) \\ &= \tan^{-1} \left(\frac{2 \times \frac{3}{4}}{1 - \frac{9}{16}} \right) - \tan^{-1} \left(\frac{17}{31} \right) \quad (1) \end{aligned}$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right]$$

$$= \tan^{-1} \left(\frac{3/2}{7/16} \right) - \tan^{-1} \left(\frac{17}{31} \right)$$

$$= \tan^{-1} \left(\frac{24}{7} \right) - \tan^{-1} \left(\frac{17}{31} \right)$$

$$= \tan^{-1} \left(\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \times \frac{17}{31}} \right) \quad (1)$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right]$$

$$= \tan^{-1} \left(\frac{24 \times 31 - 17 \times 7}{7 \times 31 + 24 \times 17} \right)$$

$$= \tan^{-1} \left(\frac{744 - 119}{217 + 408} \right) = \tan^{-1} \left(\frac{625}{625} \right)$$

$$= \tan^{-1} (1)$$

$$= \tan^{-1} \left(\tan \frac{\pi}{4} \right) \quad \left[\because 1 = \tan \frac{\pi}{4} \right]$$

$$= \frac{\pi}{4} = \text{RHS}$$

$$\left[\because \text{principal value branch of } \tan^{-1} x \text{ is } \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right] \quad (2)$$

58. Solve for x ,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3},$$

$$-1 < x < 1.$$

Delhi 2011C; HOTS

💡 Firstly, write $\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ by applying formula $\cot^{-1}x = \tan^{-1}\frac{1}{x}$ and then proceed further.

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}, \quad -1 < x < 1$$

We know that, $\cot^{-1}x = \tan^{-1}\frac{1}{x}$, so by using

this result, we may write

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad (1/2)$$

Then, given equation becomes

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \quad (1/2)$$

$$\Rightarrow 2 \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \quad (1/2)$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{2x}{1-x^2} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow 2\sqrt{3}x = 1 - x^2 \quad (1)$$

$$\Rightarrow x^2 + 2\sqrt{3}x - 1 = 0$$

$$-2\sqrt{3} \pm \sqrt{12 + 4}$$

$$\Rightarrow x = \frac{\dots}{2}$$

$$[\because \text{we know that, } x = \frac{-b \pm \sqrt{D}}{2a} \text{ where,}$$
$$D = b^2 - 4ac]$$


$$\Rightarrow x = \frac{-2\sqrt{3} \pm 4}{2} = \frac{4 - 2\sqrt{3}}{2} \text{ or } \frac{-4 - 2\sqrt{3}}{2} \quad (1)$$

$$\Rightarrow x = 2 - \sqrt{3} \text{ or } -(2 + \sqrt{3})$$

But it is given that $-1 < x < 1$, so $x = -(2 + \sqrt{3})$ is rejected. Hence, $x = 2 - \sqrt{3}$ (1/2)

59. Prove that

$$\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), x \in (0, 1). \quad \text{Delhi 2010; HOTS}$$

 Put $\sqrt{x} = \tan \theta \Rightarrow \theta = \tan^{-1} \sqrt{x}$
and then use $\cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

To prove, $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), x \in (0, 1)$

$$\text{RHS} = \frac{1}{2} \cos^{-1} \left[\frac{1 - (\sqrt{x})^2}{1 + (\sqrt{x})^2} \right] \quad (1)$$

On substituting $\sqrt{x} = \tan \theta$, we get

$$\text{RHS} = \frac{1}{2} \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \quad (1)$$

$$= \frac{1}{2} \cos^{-1} (\cos 2\theta) \quad (1)$$

$$\left[\because \frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos 2A \right]$$

$$= \frac{1}{2} (2\theta) = \theta \quad [\because \cos^{-1} (\cos \theta) = \theta]$$

$$= \tan^{-1} \sqrt{x} \quad [\because \theta = \tan^{-1} \sqrt{x}] \quad (1)$$

= LHS **Hence proved.**

Alternate Method

To prove, $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right)$, $x \in (0,1)$

$$\text{LHS} = \tan^{-1} \sqrt{x} = \frac{1}{2} (2 \tan^{-1} \sqrt{x})$$

$$= \frac{1}{2} \times \cos^{-1} \left[\frac{1 - (\sqrt{x})^2}{1 + (\sqrt{x})^2} \right] \quad (2)$$


$$\left[\because 2 \tan^{-1} x = \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) \right]$$

$$= \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) = \text{RHS} \quad (2)$$

Hence proved.

60. Prove that

$$\cos^{-1} \left(\frac{12}{13} \right) + \sin^{-1} \left(\frac{3}{5} \right) = \sin^{-1} \left(\frac{56}{65} \right) \quad \text{Delhi 2010}$$

 Use the relation $\sin^2 x + \cos^2 x = 1$. Find the value of $\sin x$, $\cos x$, $\sin y$ and $\cos y$, then use relation $\sin(x + y) = \sin x \cos y + \cos x \sin y$

$$\text{Let } \cos^{-1} \frac{12}{13} = x \text{ and } \sin^{-1} \frac{3}{5} = y$$

$$\Rightarrow \cos x = \frac{12}{13} \text{ and } \sin y = \frac{3}{5} \quad (1)$$

$$\begin{aligned} \therefore \sin^2 x &= 1 - \cos^2 x \\ &= 1 - \left(\frac{12}{13}\right)^2 \quad [\because \sin^2 \theta + \cos^2 \theta = 1] \end{aligned}$$

$$= 1 - \frac{144}{169} = \frac{25}{169} \Rightarrow \sin x = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\text{and } \cos^2 y = 1 - \sin^2 y = 1 - \left(\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow \cos y = \sqrt{\frac{16}{25}} = \frac{4}{5} \quad (1)$$

$$\text{Now, } \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\begin{aligned} \Rightarrow \sin(x + y) &= \left(\frac{5}{13} \times \frac{4}{5}\right) + \left(\frac{12}{13} \times \frac{3}{5}\right) \\ &= \frac{20}{65} + \frac{36}{65} = \frac{56}{65} \quad (1) \end{aligned}$$

$$\Rightarrow \sin(x + y) = \frac{56}{65} \Rightarrow x + y = \sin^{-1} \left(\frac{56}{65}\right)$$

$$\Rightarrow \cos^{-1} \left(\frac{12}{13}\right) + \sin^{-1} \left(\frac{3}{5}\right) = \sin^{-1} \left(\frac{56}{65}\right)$$

$$\left[\because x = \cos^{-1} \left(\frac{12}{13}\right) \text{ and } y = \sin^{-1} \left(\frac{3}{5}\right) \right] \quad (1)$$

Hence proved.

61. Prove that

$$\tan^{-1} x + \tan^{-1} \left(\frac{2x}{1-x^2} \right) = \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right)$$

All India 2010

$$= 3\theta = 3 \tan^{-1} x \quad [\because \theta = \tan^{-1} x] \quad (1)$$

$$= \tan^{-1} x + 2 \tan^{-1} x$$

$$= \tan^{-1} x + \tan^{-1} \left(\frac{2x}{1-x^2} \right) \quad (1)$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right] = \text{LHS}$$

Hence proved.

62. Prove that $\cos [\tan^{-1} \{ \sin (\cot^{-1} x) \}] = \sqrt{\frac{1+x^2}{2+x^2}}$.

All India 2010

To prove $\cos [\tan^{-1} \{ \sin (\cot^{-1} x) \}] = \sqrt{\frac{1+x^2}{2+x^2}}$

$$\text{LHS} = \cos [\tan^{-1} \{ \sin (\cot^{-1} x) \}]$$

$$\text{Let } \cot^{-1} x = \theta \Rightarrow x = \cot \theta \quad (1/2)$$

$$\text{Then, LHS} = \cos [\tan^{-1} (\sin \theta)]$$

$$= \cos \left[\tan^{-1} \left(\frac{1}{\operatorname{cosec} \theta} \right) \right] \quad (1/2)$$

$$\left[\because \operatorname{cosec} \theta = \frac{1}{\sin \theta} \right]$$

$$= \cos \left[\tan^{-1} \left(\frac{1}{\sqrt{1+\cot^2 \theta}} \right) \right]$$

$$[\because \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta]$$

$$= \cos \left[\tan^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) \right] \quad [\because \cot \theta = x]$$

(1)

$$\begin{aligned}
 \text{where, } \tan^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) &= \phi \text{ or } \tan \phi = \frac{1}{\sqrt{1+x^2}} \\
 &= \frac{1}{\sec \phi} \quad \left[\because \sec \theta = \frac{1}{\cos \theta} \right] \\
 &= \frac{1}{\sqrt{1+\tan^2 \phi}} \quad [\because \tan^2 \theta + 1 = \sec^2 \theta] \\
 &= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} \quad \left[\because \tan \phi = \frac{1}{\sqrt{1+x^2}} \right] \quad (1) \\
 &= \frac{1}{\sqrt{\frac{1+x^2+1}{1+x^2}}} = \sqrt{\frac{1+x^2}{2+x^2}} = \text{RHS} \quad (1)
 \end{aligned}$$

Hence proved.

63. Solve for x , $\cos^{-1} x + \sin^{-1} \left(\frac{x}{2} \right) = \frac{\pi}{6}$.

All India 2010C

Given, $\cos^{-1} x + \sin^{-1} \left(\frac{x}{2} \right) = \frac{\pi}{6}$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{6} - \sin^{-1} \frac{x}{2}$$

$$\Rightarrow x = \cos \left(\frac{\pi}{6} - \sin^{-1} \frac{x}{2} \right)$$

$$\begin{aligned}
 \Rightarrow x &= \cos \frac{\pi}{6} \cos \left(\sin^{-1} \frac{x}{2} \right) \\
 &\quad + \sin \frac{\pi}{6} \sin \left(\sin^{-1} \frac{x}{2} \right) \quad (1)
 \end{aligned}$$

$$[\because \cos(x-y) = \cos x \cos y + \sin x \sin y]$$

$$\Rightarrow x = \frac{\sqrt{3}}{2} \cos \left(\sin^{-1} \frac{x}{2} \right) + \frac{1}{2} \cdot \frac{x}{2}$$

$$[\because \sin(\sin^{-1} \theta) = \theta]$$

$$\Rightarrow x = \frac{\sqrt{3}}{2} \cos \left(\cos^{-1} \sqrt{1 - \frac{x^2}{4}} \right) + \frac{x}{4}$$

$$\left[\because \sin^{-1} \theta = \cos^{-1} \sqrt{1 - \theta^2} \right]$$

$$\Rightarrow x = \frac{\sqrt{3}}{2} \left(\sqrt{1 - \frac{x^2}{4}} \right) + \frac{x}{4}$$

$$\Rightarrow x - \frac{x}{4} = \frac{\sqrt{3}}{2} \left(\sqrt{1 - \frac{x^2}{4}} \right)$$

$$\Rightarrow \frac{3x}{4} = \frac{\sqrt{3}}{2} \left(\sqrt{1 - \frac{x^2}{4}} \right) \quad (1)$$

On squaring both sides, we get

$$\frac{9x^2}{16} = \frac{3}{4} \left(1 - \frac{x^2}{4} \right) \Rightarrow \frac{3}{4} x^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow \frac{3}{4} x^2 + \frac{x^2}{4} = 1 \Rightarrow \frac{4x^2}{4} = 1$$

$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \quad (1)$$

But $x = -1$, do not satisfy the given equation.

Hence, $x = 1$. (1)

64. Prove that $2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$.

All India 2010C

To prove $2 \tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right) = \frac{\pi}{4}$

LHS = $2 \tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{1}{7} \right)$... (i)

We know that, $2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

Using this identity, we can write

$$2 \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left[\frac{2 \times \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2}\right] = \tan^{-1}\left(\frac{2/3}{1 - \frac{1}{9}}\right)$$
$$\Rightarrow 2 \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right) \quad (1\frac{1}{2})$$

On putting the value of $2 \tan^{-1}\left(\frac{1}{3}\right)$ in Eq.(i),

we get

$$\text{LHS} = \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$
$$= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}}\right) = \tan^{-1}\left(\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right)$$
$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \right]$$
$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right) = \tan^{-1}(1) \quad (1\frac{1}{2})$$

\therefore The principal value branch of $\tan^{-1}x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \text{LHS} = \tan^{-1}\left(\tan \frac{\pi}{4}\right) = \frac{\pi}{4} = \text{RHS} \quad (1)$$

Hence proved.

65. Solve for x ,

$$\tan^{-1} \frac{x}{2} + \tan^{-1} \frac{x}{3} = \frac{\pi}{4}, \sqrt{6} > x > 0. \text{ Delhi 2010C}$$

$$\text{Given, } \tan^{-1} \frac{x}{2} + \tan^{-1} \frac{x}{3} = \frac{\pi}{4}, \sqrt{6} > x > 0$$

$$\Rightarrow \tan^{-1} \left(\frac{\frac{x}{2} + \frac{x}{3}}{1 - \frac{x^2}{6}} \right) = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \quad (1\frac{1}{2})$$

$$\Rightarrow \frac{\frac{3x+2x}{6-x^2}}{6} = \tan \frac{\pi}{4} \Rightarrow \frac{5x}{6-x^2} = 1 \left[\because \tan \frac{\pi}{4} = 1 \right]$$

$$\Rightarrow 5x = 6 - x^2 \Rightarrow x^2 + 5x - 6 = 0$$

$$\Rightarrow x^2 + 6x - x - 6 = 0$$

$$\Rightarrow x(x+6) - 1(x+6) = 0$$

$$\Rightarrow (x-1)(x+6) = 0$$

$$\Rightarrow x = 1 \text{ or } -6 \quad (1\frac{1}{2})$$

But it is given that, $\sqrt{6} > x > 0 \Rightarrow x > 0$

$\therefore x = -6$ is rejected. Hence, $x = 1$ (1)

66. Solve for x ,

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1} \left(\frac{8}{79} \right), x > 0.$$

Delhi 2010C

💡 Apply $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$ in LHS of given equation and then proceed further to obtain the desired result.

Given equation is

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1} \left(\frac{8}{79} \right), x > 0$$

$$\Rightarrow \tan^{-1} \left[\frac{(x+2) + (x-2)}{1 - (x+2)(x-2)} \right] = \tan^{-1} \left(\frac{8}{79} \right) \quad (1\frac{1}{2})$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right]$$

$$\Rightarrow \left[\frac{2x}{1 - (x^2 - 4)} \right] = \frac{8}{79} \quad (1/2)$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \frac{2x}{5 - x^2} = \frac{8}{79} \Rightarrow \frac{x}{5 - x^2} = \frac{4}{79}$$

$$\Rightarrow 79x = 20 - 4x^2$$

$$\Rightarrow 4x^2 + 79x - 20 = 0$$

$$\Rightarrow 4x^2 + 80x - x - 20 = 0$$

$$\Rightarrow 4x(x+20) - 1(x+20) = 0$$

$$\Rightarrow (4x-1)(x+20) = 0$$

$$\Rightarrow x = \frac{1}{4} \text{ or } -20 \quad (1)$$

But it is given that, $x > 0$

$\therefore x = -20$ is rejected.

Hence, $x = 1/4$ (1)

67. Prove that

$$\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$$

Delhi 2009

$$\text{To prove } \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right) = \frac{\pi}{2}$$

$$\text{LHS} = \sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$= \sin^{-1}\left(\frac{4}{5} \sqrt{1 - \frac{25}{169}} + \frac{5}{13} \sqrt{1 - \frac{16}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$[\because \sin^{-1} x + \sin^{-1} y = \sin^{-1}(x \sqrt{1 - y^2} + y \sqrt{1 - x^2})]$$

(1)

$$= \sin^{-1}\left(\frac{4}{5} \times \sqrt{\frac{144}{169}} + \frac{5}{13} \times \sqrt{\frac{9}{25}}\right) + \sin^{-1}\left(\frac{16}{65}\right)$$

$$\begin{aligned}
&= \sin^{-1} \left[\left(\frac{4}{5} \times \frac{12}{13} \right) + \left(\frac{5}{13} \times \frac{3}{5} \right) \right] + \sin^{-1} \left(\frac{16}{65} \right) \\
&= \sin^{-1} \left(\frac{48}{65} + \frac{15}{65} \right) + \sin^{-1} \left(\frac{16}{65} \right) \\
&= \sin^{-1} \left(\frac{63}{65} \right) + \sin^{-1} \left(\frac{16}{65} \right) \quad (1)
\end{aligned}$$

$$\begin{aligned}
&= \sin^{-1} \left[\frac{63}{65} \sqrt{1 - \left(\frac{16}{65} \right)^2} + \frac{16}{65} \sqrt{1 - \left(\frac{63}{65} \right)^2} \right] \\
&[\because \sin^{-1} x + \sin^{-1} y = \sin^{-1} (x \sqrt{1-y^2} + y \sqrt{1-x^2})] \\
&= \sin^{-1} \left(\frac{63}{65} \sqrt{\frac{4225 - 256}{4225}} + \frac{16}{65} \sqrt{\frac{4225 - 3969}{4225}} \right) \\
&= \sin^{-1} \left(\frac{63}{65} \times \sqrt{\frac{3969}{4225}} + \frac{16}{65} \times \sqrt{\frac{256}{4225}} \right) \\
&= \sin^{-1} \left(\frac{63}{65} \times \frac{63}{65} + \frac{16}{65} \times \frac{16}{65} \right) \\
&= \sin^{-1} \left(\frac{3969 + 256}{4225} \right) = \sin^{-1} \left(\frac{4225}{4225} \right) = \sin^{-1}(1)
\end{aligned}$$

\therefore The principal value branch of $\sin^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$.

$$\therefore \text{LHS} = \sin^{-1} \left(\sin \frac{\pi}{2} \right) = \frac{\pi}{2} = \text{RHS} \quad (1)$$

Hence proved.

Alternate Method Given equation,

$$\begin{aligned}
&\sin^{-1} \left(\frac{4}{5} \right) + \sin^{-1} \left(\frac{5}{13} \right) + \sin^{-1} \left(\frac{16}{65} \right) = \frac{\pi}{2} \quad \dots(i) \\
\Rightarrow &\sin^{-1} \left(\frac{4}{5} \right) + \sin^{-1} \left(\frac{5}{13} \right) = \frac{\pi}{2} - \sin^{-1} \left(\frac{16}{65} \right) \\
\Rightarrow &\sin^{-1} \left(\frac{4}{5} \right) + \sin^{-1} \left(\frac{5}{13} \right) = \cos^{-1} \left(\frac{16}{65} \right) \quad \dots(ii)
\end{aligned}$$

$$\left[\because \frac{\pi}{2} - \sin^{-1} \theta = \cos^{-1} \theta \right] (1)$$

Hence, Eqs. (i) and (ii) are equivalent. Now, we have to prove Eq. (ii).

Now do same as Que 42.

68. Prove that

$$\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}.$$

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• To prove, $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$

$$\text{LHS} = \left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} \right) - \tan^{-1} \frac{8}{19}$$

$$= \tan^{-1} \left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{9}{20}} \right) - \tan^{-1} \frac{8}{19}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \right] \text{(1)}$$

$$= \tan^{-1} \left(\frac{27/20}{11/20} \right) - \tan^{-1} \frac{8}{19}$$

$$= \tan^{-1} \left(\frac{27}{11} \right) - \tan^{-1} \left(\frac{8}{19} \right) \quad \text{(1/2)}$$

$$= \tan^{-1} \left(\frac{\frac{27}{11} - \frac{8}{19}}{1 + \frac{27}{11} \times \frac{8}{19}} \right) \quad \text{(1)}$$

$$= \tan^{-1} \left(\frac{\frac{513 - 88}{209}}{\frac{209 + 216}{209}} \right) = \tan^{-1} \left(\frac{425}{209} \times \frac{209}{425} \right) \text{(1/2)}$$

$$= \tan^{-1}(1)$$

\therefore The principal value branch of $\tan^{-1} x$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$.

$$\therefore \text{LHS} = \tan^{-1} \left[\tan \left(\frac{\pi}{4} \right) \right] = \frac{\pi}{4} \quad \text{(1)}$$

= RHS

Hence proved.

69. Prove that

$$\begin{aligned} & \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) \\ & = \frac{\pi}{4}. \quad \text{All India 2009C; Delhi 2008, 2008C} \end{aligned}$$



Apply the identity,

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right) \text{ in first two}$$

terms and the last two terms of LHS and then apply the same identity again to get the RHS.

$$\begin{aligned} \text{To prove } & \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) \\ & + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \text{LHS} & = \left[\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{5}\right) \right] \\ & + \left[\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{8}\right) \right] \end{aligned}$$

On applying the identity

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right), \text{ we get}$$

$$\text{LHS} = \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{5}}{1 - \frac{1}{15}} \right) + \tan^{-1} \left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{56}} \right) \quad (1/2)$$

$$= \tan^{-1} \left(\frac{8/15}{14/15} \right) + \tan^{-1} \left(\frac{15/56}{55/56} \right)$$

$$= \tan^{-1} \left(\frac{4}{7} \right) + \tan^{-1} \left(\frac{3}{11} \right)$$

$$= \tan^{-1} \left(\frac{\frac{4}{7} + \frac{3}{11}}{1 - \frac{4}{7} \times \frac{3}{11}} \right) = \tan^{-1} \left(\frac{\frac{44 + 21}{77}}{\frac{77 - 12}{77}} \right) \quad (1)$$

$$= \tan^{-1} \left[\frac{65/77}{65/77} \right] = \tan^{-1} (1) \quad (1)$$

\therefore The principal value branch of $\tan^{-1} x$ is

$$\left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

$$\therefore \text{LHS} = \tan^{-1} \left[\tan \left(\frac{\pi}{4} \right) \right] = \frac{\pi}{4} = \text{RHS} \quad (1/2)$$

70. Solve for x ,

$$\tan^{-1} \left(\frac{x-1}{x-2} \right) + \tan^{-1} \left(\frac{x+1}{x+2} \right) = \frac{\pi}{4} \quad \text{Delhi 2009C}$$

Do same as Que 38. [Ans. $\pm 1/2$]

71. Solve for x ,

$$\tan^{-1} \left(\frac{1+x}{1-x} \right) = \frac{\pi}{4} + \tan^{-1} x, \quad 0 < x < 1. \quad \text{Delhi 2008C}$$

The given equation is

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x$$

$$\Rightarrow \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1} x = \frac{\pi}{4} \quad (1/2)$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{1+x}{1-x} - x}{1 + \left(\frac{1+x}{1-x}\right) \cdot x}\right] = \frac{\pi}{4}$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \right] (1/2)$$

$$\Rightarrow \frac{1+x-x+x^2}{1-x+x+x^2} = \tan \frac{\pi}{4} \quad (1)$$

$$\Rightarrow \frac{1+x^2}{1+x^2} = 1 \Rightarrow 1=1 \quad \left[\because \tan \frac{\pi}{4} = 1 \right]$$

Hence, the given equation has many solutions.

(1)